Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS
IN JUNIOR AND SENIOR HIGH SCHOOLS

VOLUME XVI DECEMBER, 1923 NUMBER 6

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Published by the
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
NEW YORK CITY

Entered as second-class matter November 16, 1921, at the post office at Teachers College, New York City, under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized September 27, 1921.

THE MATHEMATICS TEACHER

OFFICIAL JOURNAL OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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[illegible] interested in mathematics and mathematics teaching are [illegible] to membership. All members receive the official journal of the Council—THE MATHEMATICS TEACHER—which appears [illegible] June, July, August and September.

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PRICE, FIFTY CENTS A COPY; $5.00 A YEAR (eight numbers)

THE MATHEMATICS TEACHER

VOLUME XVI DECEMBER, 1923 NUMBER 8

THE CONFUSION OF OBJECTIVES IN SECONDARY MATHEMATICS[1]

By WILLIAM BETZ
East High School, Rochester, N. Y.

Introduction. It is the purpose of this study to consider the transforming influence of recent curriculum theories in the field of secondary mathematics. These theories are vitally affecting mathematical instruction from a standpoint of both quantity and quality. Throughout the discussion, an acquaintance with the older literature of educational aims and values is presupposed. For the sake of brevity, moreover, the familiar, traditional point of view in the teaching of mathematics will be referred to only in a prefatory way.

Many reasons might be given for the continuous educational unrest of our age. Naturally, educational aims are always vitally affected by changed social and economic conditions. Society is never static, especially in a democracy. But aside from this permanent cause of educational readjustment, two factors have operated to bring about the present urgent demand for a restatement of educational objectives.

The *first* factor is the *measurement movement* in education. It is chiefly responsible for quantitative changes in our curricula.

The *second* factor is the enormous increase in high school attendance, more than 700% since 1890. The high school has opened its door to "all the children of all the people." In the opinion of many educators, the old academic curriculum no longer meets the needs of all these applicants. Many substitutes have been suggested. All of them have a tendency to affect radically not merely the quantity, but also the quality of high school instruction.

Let us consider these two factors in greater detail.

The measurement movement in education is hailed by some of its admirers as the most significant contribution of our age. Cer-

[1]This paper is based on a report presented at a recent meeting of the Mathematics Club of the teachers of mathematics in Rochester and vicinity.

tainly its importance can hardly be overestimated. With the aid of standardized individual or group tests, which are being improved from year to year, the acquired intelligence of children is being ascertained. Efficiency experts are developing norms by which pupils may be classified according to their determined intelligence levels.

Intelligence tests have led to two startling discoveries. (1) A surprisingly large number of children and adults are mentally "slow". (2) There is considerable evidence that the intelligence quotient of most children is constant; in other words, "slowness" is a permanent trait and follows a child throughout his school career.

Evidently, a "slow" child cannot do as much brain work in a given time as a normal child. His program must be reduced until he is equal to the task confronting him. And so educational committees have worked frantically to discover the "minimum essentials" in each subject. Each succeeding year has witnessed the formulation of ever receding minima, and the end is not yet.

The cheering counterpart of the retarded, dull child is the happy owner of a high "I. Q.". Such accelerated children, according to our best educational theory, should have an "enriched curriculum." But how shall we find time for such "enrichment" in our crowded daily calendar of recitations and school activities? Very easy, says the theorist. "Economy of time", is the patent remedy prescribed.

Any school which is fortunate enough to be able to organize and maintain homogeneous ability groups can adapt its courses to the varying mental capacities of its pupils. The situation becomes interesting, however, as soon as administrative considerations cause the usual mixture or fusion of different ability groups. If the luckless teacher tries to continue the practice of simultaneous dilution for the weak child and enrichment for the genius, he has excellent prospects of becoming a permanent inmate of a sanatorium.

This difficulty would at first seem to be a purely administrative one. Unfortunately, owing to the very large number of retarded children involved, it appears to be a case of the tail wagging the dog. Experience shows that a "minimum syllabus", intended originally for a "minimum intelligence", has a fatal

tendency to become the *real* standard for the whole school system. It is no longer considered necessary or wise to "fail" a child. He must always be given "another chance". Nevertheless, our classes are forever glutted with "repeaters". Who is to blame? Teachers and administrators are, after all, only human. And so we find that in response to a real or alleged nation-wide demand, the "slow child" has become the actual arbiter of our educational machinery.

The simultaneous insistence on minimum essentials and on greater economy of time has become a source of great confusion. Unquestionably, it has had a very marked effect, often beneficial, on courses of study and on textbooks. Each school subject has had to face a rigid inspection. Time-honored heirlooms have been swept away, and new syllabuses have been written which reflect a desire to retain only fundamentally important materials. Thus it is that the *quantitative* aspect of each school subject is being adjusted in accordance with new criteria of selection.

In this process, mathematics is receiving its due share of attention. Arithmetic has assumed more modest dimensions in the intermediate grades. The old percentage mania is being relieved by work in intuitive geometry and by simple algebraic exercises. Algebra in the ninth year has been both reduced and enriched. The terrors of factoring and complex fractions have almost vanished. Quadratics, exponents, and radicals, along with other "high-mortality topics", have been rendered safe and harmless. Instead, there is an increasing exposure to formulas, graphs, numerical trigonometry, and other aspects of functional thinking. Demonstrative geometry, whether begun in the ninth or in the tenth year, has enjoyed a similar fate. The idea seems to be that the number of theorems proved is immaterial, since the system as a whole is not rigorous anyway. The present plan, endorsed by various important examining bodies, offers a very generous list of assumptions, followed by a shrinking body of propositions. The time thus gained is intended to make possible numerous interesting applications, or—in the case of brilliant pupils—a necessarily brief exposure to solid geometry.

It is rumored that certain private schools actually boast of accelerated classes in geometry.

Now, these quantitative readjustments might be entirely salutary, if it were not for the increasing adoption of minimum standards of promotion as a natural consequence of minimum achievement in each subject. In many classes, according to reliable evidence, it is no longer a case of "a few things well done", but rather a case of even these few things done very poorly. And yet a cumulative subject like mathematics, when given such a weak and anaemic foundation, cannot function either in a practical or in a cultural direction.

So far, we have considered exclusively the quantitative aspect of curriculum reconstruction. The remainder of this report is to be devoted primarily to the *qualitative* side of the question.

Teachers of mathematics have cooperated, often enthusiastically, in the reorganization of their subject along the lines indicated above. This is not true to nearly the same extent of certain more radical attempts at reconstruction which are to be pointed out below, and which are adding to the prevailing confusion of objectives. It is here that non-mathematicians, usually administrators or professors of education, have played a leading role. They have offered suggestions or extended programs which range all the way from the complete elimination of secondary mathematics in the lives of many children to a very specialized and greatly reduced offering of the subject. The arguments advanced by these reformers are very similar and not without plausibility. Vocationalism and socialization represent the dual shibboleth of this group of thinkers. School children are pictured by them as possessed of definite, final life-career interests. These sacred interests must be safeguarded. And so we witness the demand for a still more refined differentiation of pupils, not based on ability, but on real or imaginary specific objectives.

There is nothing abnormal, or even original, about the idea of emphasizing the really essential parts of any school subject in the case of any child. And it is entirely probable that for certain groups of pupils the brief exposure to mathematics contemplated by some educators has no particular value. Many of these children would receive greater benefit from continued attention to the fundamentals.

However, the visionary character of curriculum criteria based *exclusively* on vocational or social interests is almost too appar-

ent to merit discussion. One thing is quite certain. It makes a vast difference whether a subject is studied for its own sake, or is merely the handmaid of a trade. In one case we study *principles* which owe their validity to the insight of the learner; in the other case we manipulate *rules* which are not derived but imposed. The creation of a "consumer's mathematics," "devoid of technicalities and dead formalism," appeals to the popular preference for ease and comfort. But one has only to listen to the tales of woe of shop instructors, contractors, managers of engineering offices, and even of business men, to realize that a "thin" course in mathematics, based on a meager collection of half-assimilated formulas and rules, is not a safe and reliable life equipment.

In the following pages a few characteristic utterances on mathematical reform by well-known, progressive educators will be submitted. They were chosen almost at random, and might easily be extended indefinitely. By way of preface, and also of contrast, the sane position of the National Committee on Mathematical Requirements will be considered first.

1. *Attitude of the National Committee.* In Chapter II of its report on "The Reorganization of Mathematics in Secondary Education," the National Committee on Mathematical Requirements attempted to lay down some general principles which should govern the aims of mathematical instruction. Only "general" aims are considered in the report. These are described as (1) practical, (2) disciplinary, and (3) cultural. An admirable discussion of these three types of aims is given on pages 5 to 8 of the report. Every educated person, says the National Committee, is vitally affected by these three types of aims. The point of view which should govern instruction is formulated by the National Committee in the following cardinal sentence: "The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual." It is further asserted that "drill in algebraic manipulation should be limited to those processes and to the degree of complexity re-

quired for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take." Finally, the National Committee expresses the opinion that "in the junior high school, comprising grades seven, eight and nine, the course for these three years should be planned as a unit with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little reference to courses which he may or may not take in succeeding years." In other words, the National Committee seems to believe that the work of each grade can be so organized that it is a worth-while unit in itself and at the same time is preparatory for subsequent work. In view of the well-known cumulative character of mathematics, it is at least a debatable question whether courses can be so arranged for *each* year as to be both "terminal" and preparatory. Evidently, this matter requires further investigation. Moreover, the exclusive consideration of "general aims" by the National Committee leaves open the important issue of the specific mathematical needs of vocational and technical groups.

2. *Specific Objectives of the Kilpatrick Committee.* (See Bureau of Education Bulletin, 1920, No. 1, "The Problem of Mathematics in Secondary Education," report of the commission on the reorganization of secondary education appointed by the National Education Association, Washington, 1920.) Professor Kilpatrick's committee attempted to take due account of individual differences and to ascertain appropriate objectives for various groups of pupils. The findings of the committee may be inferred from the following quotations:

> "The fact of marked individual differences has been scientifically established. The principles of adaptation to such individual differences, that is, to individual needs and capacities, is now widely accepted in the high schools of America. The exception calls for scrutiny. Traditionally, algebra and geometry have been required for graduation. Is this necessary or advisable?"

The committee considers that four groups of users of mathematics may be distinguished:

> (a) The "general readers" who will find their use of mathematics beyond arithmetic confined largely to the interpretative function described above.

(b) Those who work in certain trades will make limited, but still specific, demand for the "practical" use of mathematics.

(c) Those whose practical work as engineers or as students of certain sciences requires considerable knowledge of mathematics.

(d) Those who specialize in the study of mathematics with a view either to research, or to teaching, or to the mere satisfaction of extended study in the subject."

The committee then outlines, somewhat in detail, the alleged needs of the several groups. The position of the committee in the actual construction of specific curricula is shown in the following statement (p. 15):

> Out of the conflict of topics for a place in the program there emerges one general principle, already suggested in these pages, which is being increasingly accepted for guidance by students of education. In briefest negative terms: No item shall be retained for any specific group of pupils unless, in relation to other items and to time involved, its (probable) value can be shown. So stated, the principle seems a truism, but properly applied it proves a grim pruning hook to the dead limbs of tradition. A final method of ascertaining such comparative values remains to be worked out; but the feasibility of a reasonable application of the principle will hardly be denied. In accordance with this, many topics once common have been dropped from the curriculum and more are marked to go.

Should one feel inclined to accept all of the recommendations of this committee, the almost insuperable difficulty of carrying them out in actual practice would remain. The writer knows of no public school system which, under existing conditions, can introduce differentiated courses similar to the ones suggested by the committee. Even in cities which provide very generously for "guidance" it has usually been impossible to classify pupils into highly differentiated groups, beyond the ordinary classification along academic, technical, and commercial lines. The fact remains that only a minority of pupils in the high school have very clear notions as to their future careers. In small towns, all attempts at creating "life-career groups" are out of the question anyway.

3. *Vocational Objectives vs. General Training.* How far a noted educator may be led to espouse one single aspect of the

educational problem is illustrated conspicuously by Professor F. Bobbitt's attempt to devise definite criteria for the construction of special curriculums. In a recent monograph ("Curriculum-Making in Los Angeles," University of Chicago, 1922) he describes, in detail, how each course of study is to be reorganized in accordance with his elaborate body of social and vocational objectives. The entire chapter on mathematics (Chapter VII) should be quoted at this point in order to convey a correct notion of the author's extreme position. Professor Bobbitt loses no opportunity to discredit the alleged disciplinary values of arithmetic, algebra and geometry. The objectives of the National Committee are characterized as "purely academic." With regard to the report of the National Committee, Professor Bobbitt says: "As one reads the pages, one feels one's self wholly within an academic atmosphere, and never at any time does he get a real whiff of the world's actual life, and of the mathematics that actually functions in the real lives of men and women." Professor Bobbitt's chief contentions are embodied in certain statements which were issued to the teachers of Los Angeles at the beginning of his investigation, and also in certain basal assumptions, which, in his opinion, should underlie a general course in mathematics. The following quotations characterize his point of view:

> It seems that the department of mathematics in the high schools has looked rather less than almost any other department to the actual social needs of men and women. It has found the usual high school mathematics to be of vocational service to a minority of the high school population, and on this vocational basis tends to prescribe it for the entire high school population—a manifest absurdity, though happily diminishing in progressive school systems. Los Angeles, we are glad to say, has advanced far beyond this point. Yet much remains to be done.
>
> No department more than that of mathematics needs to make clear distinction between general training and vocational training and to organize the two entirely independently. The highly differing mathematical needs of different vocations must not be permitted to dictate the content of that general training course which has no reference to the vocation into which one goes. On the other hand, the general training in the field of mathematics must not serve as a limitation upon the mathematics prescribed for vocational purposes. In view of certain of these difficulties the

following statement was issued to the teachers of Los Angeles as they began their considerations:

It is probable that no department of the high school has a more difficult or baffling problem of reorganization of its activities than the mathematics department. Let us illustrate:

The city has decided that commercial students going into business do not need algebra, geometry, or trigonometry for general, cultural, or disciplinary training.

Since this is a large and representative group of students it appears to follow, if this decision is correct, that students in general do not need algebra, geometry, or trigonometry for general, cultural, or disciplinary training.

The city has decided that commercial students need full and intensive training in the mathematics of their vocation.

This probably typifies the need of every vocational group. It needs full and intensive training in the mathematics of its vocation. But the mathematics will differ greatly from vocation to vocation and must be administered, therefore, according to the special needs and as a part of the vocational training.

The vocational mathematics for commercial students is administered in this city as a vocational course in the commercial department.

This appears to represent the proper placement of all vocational mathematics courses, not in the general department of mathematics, but in the appropriate vocational department.

It is the belief of the writer that these decisions of the city as regards the mathematics of the commercial students are educationally correct; and that the deductions that appear naturally to follow are educationally correct.

If this is true, then algebra, geometry, and trigonometry have justifiable place in the curriculum only when they are necessary portions of vocational courses; and in such cases the specific content is differently dictated by different vocations.

These statements are equally applicable to those who finish their schooling with the high school and those who take additional years of work in college. Neither the length nor the place of one's training dictates one's needs. . . .

Basal Assumptions: The basal assumptions which should probably underlie the mathematical aspect of the general training, as stated below, are intended to be those which underlie the general, not the vocational training.

1. The mathematics to be included in the general training should be determined by what men and women actually need

in their general affairs outside of their several callings, and by the common mathematical element of all vocations.

2. The major thing needed is not ability to solve difficult mathematical problems; it is rather ability and disposition to think accurately and quantitatively in one's affairs. The latter frequently involves mathematical operations as incidental matters—never as the fundamental ones.

3. The way to learn to think quantitatively is mainly to think quantitatively in those various fields where quantitative thought is possible and desirable.

4. The ability to do quantitative thinking is to be developed in youth under conditions as nearly like those in which it is to function in adulthood as practicable.

5. The ability to think quantitatively is a general need. It should, therefore, be a portion or aspect of the required training.

6. While the mathematical operations are not the main things, yet it is indispensable that one perform the needed ones with certainty and skill.

7. Outside of their vocations, the citizens of Los Angeles do not use algebra, demonstrational geometry, or trigonometry.

8. Outside of their vocations, the only mathematics content really needed by the men and women of the city is applied arithmetic.

9. Even in their vocations, only a small percentage of the citizens of Los Angeles use algebra or trigonometry; and practically none use demonstrational geometry.

10. The mathematics needed for one's vocation should be determined strictly with a view to that vocation. It should then be administered only to those who enter that vocation; and it should be very thorough, especially along the applied lines involved in that vocation.

11. As fields of intellectual play, neither algebra nor demonstrational geometry lay foundations or centers of systems of ideas and thought generally needed throughout life.

12. As matters of pure general discipline, the city cannot afford to administer algebra and geometry purely on faith; the specific disciplinary values should be made clear; and it should be demonstrated that they are or can be attained.

13. The value of applied mathematics, intensive and thorough, for producing power to think, to assemble and organize facts, etc., has been amply demonstrated.

14. The content of the economic and civic studies needs to be developed so as to include the necessary large amount of applied mathematics.

15. The mathematical element of the science studies, particularly general science and the biological sciences, needs much further development.

16. The needed mastery of the world of number is to be attained mainly through using number—not by studying abstractions about number.

17. The needed mastery of the world of form and space-relations is to be attained mainly by using forms and by constructing forms that are to be used. Studies about forms need to be only brief and incidental.

How anyone acquainted with actual high school conditions and with the principles of democratic American education can tolerate the views of Professor Bobbitt, is a mystery to the writer. And yet it is precisely these one-sided proposals for the solution of our educational problems which are disturbing the educational balance of many school administrators and lead to ill-considered and premature readjustments.

4. *Correlation Between Aims and Activities.* Professor A. Inglis of Harvard University is known as an advocate of a proper correlation between aims and activities. In a recent study published in "Educational Administration and Supervision," December, 1922, Professor J. A. Clement of Northwestern University outlines that attitude as follows:

> Several years ago Professor Inglis wrote that "the key to any analysis of aims in education is to be found in an analysis of the activities of life in which people do or should engage." There are three groups of activities involved:
>
> (1) Those relating to participation in citizenship, included in the social-civic aim;
>
> (2) those relating to economic production and distribution, included in the economic-vocational aim;
>
> (3) those relating to free use of leisure time, included in the individualistic-avocational aim. As is true in the instances of afore-named aims according to other writers there is here an interrelation of all of these three aspects in the generic social aim of education.
>
> All of these three aims together with their corresponding functions, adjustive, integrating, differentiating, propaedeutic, selective, and diagnostic, must be considered in the formulation of curriculums. The basis for a single line curriculum, or other types and the bases for differentiated curriculums must be consciously considered.
>
> It is necessary in the light of such aims to determine what subjects shall be included for the pupils, as Inglis suggests.

> . . . In addition to determining what subjects shall be commonly studied by all pupils it is necessary to re-evaluate the aims of each subject and therefore to reorganize the nature of much of the subject-matter offered under each school subject so as to conform to the redetermined aims.

Professor Inglis classifies the values of mathematical studies under the following heads:

(1) Direct and specific use values:
  (a) General use values;
  (b) Specific vocational values;
  (c) Propaedeutic values.
(2) Indirect and general values:
  (a) Conceptual values;
  (b) Transfer values.

In his "Principles of Secondary Education," Professor Inglis undertakes an analysis of each of these aims and values. It is a notable contribution to the literature of objectives. One of Inglis' most significant comments deals with the function of applied mathematics. It will be referred to below. That Professor Inglis endorses differentiation, is proved by the following quotation:

> The fact that justification can be found for the teaching of secondary school mathematics on the basis of its indirect values and the fact that those values are not limited to special curriculum groups of pupils do not justify any assumption that all pupils should study algebra and geometry. The common requirement of the study of algebra and geometry on the part of all pupils is opposed to the conception that individual differences in capacities, interests, and future activities, must be recognized. This true whether we accept or reject indirect values.

5. *Sociological Determination of Objectives.* The outstanding exponent of a sociological attitude toward modern educational objectives is Professor David Snedden of Teachers College, Columbia University. His point of view is also summarized by Professor Clement in the study above referred to. Professor Clement says:

> Professor Snedden has suggested that the aims of education should be determined on the basis of sociological analyses. To quote verbatim, he writes: "We can have no satisfactory set of working principles in the construction of curricula until we possess fairly acceptable analyses, quali-

tative and quantitative, of the value of social life.'' In such analyses the relative values of health, knowledge, beauty, sociability, etc., will need to be reliably determined.

His attitude toward the determination of objectives through surveys is clearly set forth in the following words: ''It was only when objective sociological studies of the conditions surrounding the work of men and women in actual vocational practice were instituted that the promoters of vocational education found themselves on sound ground. Surveys were begun with a view to obtaining reasonably correct answers to such questions as these: What are the various vocations now followed in a given community? How many workers in such? When, where, and how did these acquire the competency they now possess? . . . For the present the situation is one of confusion. The historic studies, deeply rooted in custom, and frequently popular in approval, exhibit as yet few changes even in progressive junior high schools. The new studies encounter opposition because their actual objectives are as yet so ill-defined.

In his latest volume, entitled ''Educational Sociology'' (The Century Company, 1922), Professor Snedden devotes about twenty pages to a description of the mathematical studies. After a critical arraignment of the customary claims made for secondary mathematics, Professor Snedden says:

> The time is now ripe for a re-examination of these alleged values, particularly those of algebra and plane geometry. It should, in the first place, be an easy matter to determine the scope and character of probable vocational needs (including those of professional studies) for these subjects. It should furthermore be an easy matter to determine how far study of these subjects by methods now prevalent in our schools contributes to the realization of cultural objectives as these may be properly analyzed and defined. Present psychological theory and demonstration may not give us entirely satisfactory grounds for establishing the limits of their disciplinary values; but it is sufficient to enable us to view with much skepticism dogmatic valuations and practices entailing the prescriptive and enforced study of them. Many educators are now quite ready to place both algebra and plane geometry in the list of purely elective subjects for school and college, always with the reservation that they be prescribed for candidates seeking admission to those professional schools in which they demonstrably serve as necessary tools.

Concerning the new courses in our junior high schools, Professor Snedden asks these questions:

(a) What courses in mathematics is it practicable to offer in the curricula of large junior high schools?

(b) Should all pupils be required to take in common certain courses or portions of mathematics?

(c) Should all be required to take specified minimum amounts of mathematics, choosing, under advice, from the various courses offered?

(d) Should all mathematics courses be placed in a list of electives, to the end that pupils may, with the approval of their parents, and aided by competent educational advice, choose to take much, little, or no work in this field, provided their programs are properly complete with other valuable studies?

(e) Should certain pupils be debarred from specific courses?

Professor Snedden claims that the following courses are now practicable in any large junior high school:

(a) An advanced, fairly "stiff" two years' course in arithmetic of the type that has been largely traditional heretofore in seventh and eighth grades.

(b) The more able-minded pupils certainly could take "pure" algebra and plane geometry, slightly simplified from the standards now commonly held for these subjects in ninth and tenth grades.

(c) A one year's course in "constructive" geometry is feasible.

(d) A one or two years' course in "composite" mathematics (blending the methods of arithmetic, geometry, algebra, in their application to relatively general and realistic problems of practical life) is certainly to be deemed practicable in the light of recent studies and texts.

(e) A one year's course (or even a two years' course) in "commercial" arithmetic (or mathematics), centering largely in the problems arising in the business transactions of buying, selling, and accounting.

(f) A one year's course, centering largely in problems peculiar to general farming.

(g) A one year's course, centering largely in problems peculiar to home-making.

(h) A one or two years' course, centering largely in problems peculiar to mechanical shops and the building trades.

(i) Though no good examples yet exist, it is theoretically possible to work out a substantial one year's course in almost pure "utilizer's" mathematics beyond the standards

achieved in the first six grades, topics being restricted to those based on the common needs of large numbers of men and women in buying, reading, and modest investment.

(j) It is also theoretically possible to provide one or more courses in pure "appreciational mathematics" of a liberal or cultural nature, just as it is possible to have "appreciational" courses in poetry, music, general science, or drama. Historical materials might play a part here, as they do in appreciational courses in music, fiction, or graphic art—but surely only a minor part. True appreciation should center primarily in vital contemporary achievements; historical beginnings may be somewhat illuminating, but that is all.

(k) Finally, it is practicable to offer, in "short unit" or other form, courses in "review" arithmetic, designed primarily to keep in vital and functioning form over the "forgetting" years of childhood the "utilizers'" and fundamental arithmetic learned in the first six grades.

Professor Snedden's mathematical creed is embodied in the following conclusions:

1. The pupils attending any junior high school represent wide ranges of abilities, conditions, and educational needs.

2. The possible educational offerings of such a school are steadily increasing in number and variety. None of these are equally necessary or valuable for all pupils. Each one is probably very valuable and important for some pupils. The junior high school may be expected to repeat the history of the liberal arts college and the senior high school as respects the range of subjects that may profitably be offered. Under these conditions, obviously, the responsibility of the school in recommending, prescribing, or prohibiting to each distinctive group of learners certain studies becomes very great.

3. It is practicable and probably desirable to offer in the large junior high school a variety of mathematical courses—perhaps not fewer than eight nor more than twenty distinctive one-year courses.

4. If proper attention has been given to essential "utilizers' arithmetic" in the first six grades, it is submitted that careful study of the most urgent educational needs of the various case-groups will prove that there is no mathematical topic or course that need thereafter be prescribed for all pupils alike.

5. Subject to the one condition stated below, it is probable that careful analysis of the educational values of the large variety of different subjects that can and should be offered in the junior high school, as these minister to the respective educational needs of various case-groups, will

establish the inadvisibility of prescribing certain quantities of mathematical study, of one kind or another, for all pupils; provided that pupils of less than average ability shall, in response to tests given from time to time, demonstrate that their "utilizers'" mathematical knowledge learned in the first six grades remains actively "functional."

6. All mathematics courses in the junior high school should, therefore, be placed in an elective list from which individual pupils should be advised and helped, with the consent of parents, to select those that will most probably meet their personal and social needs.

7. Where courses are especially designed for pupils of superior abilities, it is, of course, legitimate to bar from them pupils of less than the required abilities.

(A summary of Professor Snedden's mathematical program may also be found in his "Problems of Secondary Education." Houghton Mifflin Company, 1917, Chapter XVIII.)

What shall we say to all this? Evidently, even if one should feel disposed to accept Professor Snedden's contentions in their entirety, there remains the same fatal difficulty which was referred to in previous paragraphs. It is this: financial and administrative reasons have made it absolutely impossible, even in very large junior high schools, to attempt more than a rudimentary differentiation of courses. And assuming differentiation to be theoretically possible, would the best interests of any pupil be served by his early assignment to some strictly vocational or "sociological" group?

6. *Correlation of Ideals and Activities.* The typical representative of this mode of determining the educational objectives is Professor W. W. Charters, of the Carnegie Institute of Technology.

> "The curriculum," says Professor Charters, "is derived from both ideals and activities. Virtue, swift-footedness, piety or social efficiency must be set up in a system of education. But in order to select the material to which these shall apply it is absolutely essential for the teacher to know the activities, problems, thoughts, or needs in connection with which these ideals are to operate. . . . . The curriculum by which he is trained to perform the important activities of the group in accordance with the highest ideals is necessarily based upon both activities and ideals. And obviously any aim of education expressed only in terms of ideals must fail to function."

In his recent volume on "Curriculum Construction" (MacMillan Company, 1923), Professor Charters elaborates his point of view at great length. As to the objectives of education, Professor Charters says,

> "Statements of objectives of education and of subjects of the curriculum lead to confusion unless they are analyzed into ideal objectives and acticity objectives. Social efficiency as an objective becomes easily analyzed if we see that it means social ideals and social activities."

Professor Charters says that ideals, however excellent, are not a sufficient basis for a curriculum. It is also necessary to take into account the varied activities which are important in our age.

> The social demand must be studied so as to enable the faculty to interpret the spirit of the age. To this end, the opinions of thoughtful men and women in public and private life, as expressed in speech or writing, need to be carefully weighed. Not only so, but a thoroughgoing investigation of the needs of the students who are to be instructed must be followed through. From all sources, the faculty must gather data upon which to base their judgment.

Professor Charters devotes an entire chapter to typical studies in the field of arithmetic, which were made with a view to determining more accurately the content of the curriculum from a social, commercial, and technical point of view. Similar studies would, of course, have to be made with reference to all other mathematical branches.

Criticisms and Conclusions. The impression created by the quotations submitted in the preceding pages on the mind of a critical reader will depend, of course, upon his educational orientation. Undoubtedly these passages reflect the honest convictions of their authors. For that reason they are entitled to respectful consideration. Any courageous and significant analysis of our educational difficulties should be greeted as a laudable contribution. Nor is there any objection to drastic proposals as such.

The real defect in many of these reform plans is their lack of definiteness, their complete disregard of organization problems, and their crass utilitarian philosophy which seems utterly indifferent to ultimate mental and spiritual consequences. Any pro-

posed readjustment, to be effective, must take due account of such tangible factors as school finances, administrative feasibility, and pedagogic soundness. In other words, we must have *positive* criteria of curriculum-building that will function not merely in a few highly favored institutions, but also in our crowded city schools. There must be adequate principles of selection, of arrangement, and of achievement, in each school subject.

Such criteria are gradually emerging. The topics selected must satisfy the demands of reality, of social worth and of thinking value. The content of a curriculum will seem real to a pupil only if he can interpret new materials by means of bonds already existing in his mind. "Social worth" must not be based exclusively on frequency arguments. There may be widespread but undesirable social customs and tendencies which should be discouraged or ignored by the school. If a higher cultural level is ever to be reached by our developing democracy, we must cease to appeal to "Main Street" for slogans and ultimate standards. Not what *is*, but what *should* be, must receive our constant scrutiny. Otherwise, mediocrity and stagnation will be our perpetual destiny.

Nor can we ever accept a purely "practical" type of mathematics in our public high schools. The "problem solving attitude," with its regard for thinking values, is absolutely essential. This means that the "how?" of each process must be accompanied by its "why?." Performance should not be separated from understanding and appreciation.

There must also be reliable criteria of *arrangement*. The relative difficulty of the customary mathematical topics is being ascertained by psychological analysis and by experimental teaching. However, a cumulative subject like mathematics cannot disregard logical considerations. Hence an ideal teaching sequence must be the result of a scientific compromise between insight and perspective.

Finally, we must have reliable standards of achievement. The present literature of standard tests in mathematics represents merely a point of departure. Before we can set up "norms," we must agree upon the conditions under which they are to be valid. It will make all the difference in the world whether we

are thinking of a brief "exposure course," of practical application only, or of a liberal education.

In the light of the preceding considerations it is obvious that the four sources of confusion named above, namely "minimum essentials," economy of time, vocationalism, and socialization, require a more definite interpretation with reference to selection, arrangement and achievement. It is a pleasure to note an increasing demand for tested objectives that are in harmony with common sense and with sound educational doctrine. Teachers of secondary mathematics will do well to give careful attention to the investigations of the American Classical League, conducted in cooperation with the General Education Board. The preliminary reports issued by the special investigating committee are models of searching analysis which will retain a place of honor in the literature of educational objectives. (See *Classical Journal,* June, 1923.) Reference may also be made to a recent address by Professor Thomas H. Briggs of Teachers College, Columbia University, for a clear statement of the new demands in secondary education. ("What Next in Secondary Education," *School Review,* September, 1922, pp. 521-532.)

We are now ready for the crucial question: Are there any definite plans, suggested by experience, which promise to solve the vexing problems outlined above?

Fortunately, there is good ground for an *affirmative* answer. In fact, certain schools have actually begun to face the situation with honest determination. A brief outline of the principles which will probably govern the coming reorganization of secondary mathematics will be offered as a conclusion of this study.

In the first place, the junior high school should furnish a *common* mathematical background to all pupils. The fundamental facts of arithmetic, geometry and algebra cannot be changed. There can be no effective use of mathematics, either theoretical or practical, without a mastery of its elementary tools. Hence any differentiation during the seventh, eighth, and ninth years must be quantitative, not qualitative.

Secondly, there must be no artificial separation of theory and practice in secondary mathematics. There is increasing agreement on this point. (See T. Percy Nunn, *The Teaching of Algebra,* p. 17). Mathematics is almost encyclopedic in its wealth of possible applications. Number and form are the foundation of

our entire material civilization. The examples of Archimedes, Newton, Gauss, and many others, demonstrate the fruitfulness of intelligent attention to the interaction of pure and applied mathematics. Egypt and Greece typify opposite poles to be avoided in the classroom. "Practical mathematics," as such, is helpless without the underlying theory. Moreover, it has been asserted by noted educators that pure science is of greater disciplinary value than applied science. (See Colvin, *The Learning Process*, pp. 249-250.) Professor Inglis evidently agrees with this view, for he says: "Attempts to emphasize applied mathematics do not meet the conditions favorable to extensive transfer. Valuable though applied mathematics is and important though 'real' problems may be, their function is to attain *direct* values and not to foster conditions favorable for transfer. It is a tenable thesis that for most pupils pure mathematics is superior to applied mathematics because of its greater transfer values."

Thirdly, there must be continuous reference to objective standards of classification. Ability grouping of pupils is necessary at the beginning of the seventh school year. This should be followed by another ability selection at the end of that year, based on three factors: (1) intelligence tests, (2) the complete school record of the pupil, (3) a comprehensive achievement test of the newer type in at least two subjects, preferably English and mathematics. Any pupil whose final record falls below a moderate but fixed objective standard, say sixty or seventy per cent, should be debarred from courses prescribing a study of foreign languages or advanced mathematics. The normal pupils can then complete the course in a really efficient manner, while slow pupils will advance at their own optimum rate. A similar readjustment is desirable at the end of the eighth year. The ninth year course for the slow pupils should not be college preparatory, but should include carefully chosen units of intuitive geometry, algebra, numerical trigonometry and business arithmetic.

Fourthly, qualitative differentiation is desirable at the end of the ninth year. College preparation will demand the attention of normal or accelerated groups. Slow pupils will have one of three options. They may discontinue their mathematical studies. By completing the prescribed algebra requirement they may qualify for admission to a college preparatory class. They may

also elect any one of a number of short industrial or commercial unit courses, to be completed within six months or a year, preferably in another building. These unit courses will give due attention to the specialized interests of vocational groups.

There is nothing undemocratic about such a plan. It does not close the door to any worthy pupil, but it refuses to fit square pegs into round holes. Above all, it does not sacrifice standards of achievement. Eventually, it will be found more economical than the usual incessant repetition of impossible curricula by unassorted and indifferent masses of children. It recognizes the fact that nearly all the sciences are becoming more quantitative from day to day, that practical men are among the first to deplore a weak school course in mathematics, that the traditional belief in the high cultural and disciplinary value of mathematics is not shaken. (See National Report, Chapter IX.)

No single plan will remove all the difficulties, nor can it escape the transforming influence of new conditions. Continuous effort will have to be directed toward an ever clearer formulation of tested objectives. But that is not all. A perfect musical instrument reveals its possibilities only at the touch of a master. And so, in addition to valid objectives, we must have inspired teachers and scientifically prepared textbooks. In that case we may confidently hope to retain for the most ancient and the most perfect of all the sciences the esteem which it has justly enjoyed throughout the epochs of recorded history.

SOME VARIETIES OF SPACE

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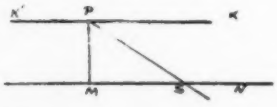
The purpose of this paper is to make a rapid survey from the standpoint of geometry of some of the more important varieties of space. Of necessity only the most striking of the simpler properties can be brought out, and any theorems given will be stated without proof. The spectacular element that comes from endowing figures in the various spaces with reason, useful as it is for illustration, will be omitted.

First let us consider space as Euclidean geometry has interpreted it to us. The elements assumed in that space are points, straight lines and planes. The following properties will serve to differentiate this space from the non-Euclidean spaces that are to be discussed.

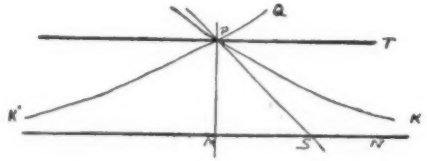
Take a straight line, MN, and a point P at a distance MP from MN, and let a line PS rotate about P in the positive direction of rotation. Then S, the point of intersection of PS with MN, moves steadily toward the right. When the angle MPS



becomes a right angle, MPK, the rotating line ceases to cut MN and we say that PK is parallel to MN, or that PK cuts MN at infinity. If the rotation takes place in the negative direction the line PK' appears as parallel to MN, but since the angle $K'PK$ is a straight angle, these two parallels form one and the same straight line. Therefore, in Euclidean space one line and only one can be drawn through a point so as to be parallel to a given line. From this property of Euclidean space we have the various theorems about parallels cut by transversals, the theorem that the sum of the angles of a triangle is a straight angle, the construction of similar triangles, and so on.

The two non-Euclidean geometries that are to be discussed here arose when geometers first followed to their logical conclusions the hypotheses that through a point more than one line could be drawn parallel to a given line, and that through a point

no line could be drawn parallel to a given line. The first gave rise to hyperbolic geometry, the second to elliptic geometry. In representing the figures of these two geometries upon the Euclidean plane it is often necessary to give an apparent curvature to a straight line in order to bring out its characteristics, but in either hyperbolic or elliptic spaces the straight lines and the planes are as free from curvature as are those in Euclidean space.



In hyperbolic space let us consider a figure similar to the one already used for Euclidean space. With a given straight line MN and a fixed point P at a distance MP from MN let a straight line through P cutting MN in S rotate about P in the positive direction. The point S moves steadily toward the right. As the rotating line takes up the position PK there is for the first time no finite intersection with MN, and in this position PK is said to be parallel to MN. By rotation in the negative direction a second position, PK' is reached in which the rotating line has no finite intersection with MN. So far there seems no difference between the state of affairs here and in the Euclidean figure considered earlier, but the important fact here is that the angle MPK is less than a right angle, and since by symmetry MPK' is equal to MPK the two lines PK and PK' are distinct, and we have the non-Euclidean possibility of being able to draw through any point two distinct lines both parallel to the same straight line. The angle MPK is called the angle of parallelism, and its size varies with the distance of P from MN. As P recedes to an indefinitely great distance from MN, the angle of parallelism becomes indefinitely small.

A few of the noteworthy properties depending upon this new theorem of parallels are the following: PK approaches its parallel MN indefinitely in the direction of parallelism but recedes from it indefinitely in the other direction; similar figures cannot exist, *i. e.*, if two triangles have their angles equal each to each, the triangles must be equal throughout; the sum of the

three angles of a triangle is less than two right angles, and is not a constant; and this space contains a triangle of maximum area with the following properties—the three vertices are all at infinity, the three vertex angles are all zero angles, the three sides are parallel in pairs.

In this hyperbolic plane there are three varieties of lines passing through the point P; first, the lines that intersect MN in ordinary points, second, the two lines parallel to MN that are said to intersect MN in points at infinity, third, lines through P that lie in the angular opening KPQ and that meet MN neither at infinity nor in the finite parts of the plane. Among the lines of this third group there is a line PT that is perpendicular to MP. Here we have another distinguishing property of hyperbolic space brought forward, for here two lines perpendicular to the same straight line neither meet nor are parallel. From this come the theorems that non-intersecting straight lines have a common perpendicular, and that this common perpendicular measures the shortest distance between these lines. On either side of the common perpendicular the lines diverge. The companion theorem for parallel lines is that two parallel lines have no common perpendicular.

Even in Euclidean geometry it is often convenient to look upon points as the vertices of pencils of lines passing through them in the plane. As all parallel lines are assumed to determine the same point at infinity, Euclidean points may be divided into two classes, those that form the vertices of pencils of convergent lines, and those that form the vertices of pencils of parallel lines. The former are points in the finite part of the plane, the latter are points at infinity. In hyperbolic geometry this conception is especially useful. Here there are three kinds of pencils each defining a geometric entity to which by analogy we may give the name point—pencils of convergent lines giving ordinary points, pencils of parallel lines giving points at infinity, and pencils of lines perpendicular to another line defining something to which is given the name ideal point. This last variety of points lies outside of all space, but the name is justified by the fact that their properties are the same as those of ordinary points, that is, any two ideal points determine a line, any three determine a circle, and so on.

The simplest curve in Euclidean geometry is the circle, which is usually defined by means of the distance of a moving point from a fixed point, but even in this geometry it is occasionally convenient to consider a circle as defined by its property of cutting all its radii at right angles, that is, a circle may be defined as the orthogonal trajectory of a pencil of lines, the vertex of the pencil being the center of the circle. Thus in the Euclidean plane there are two types of circles, those with their centers in the finite part of the plane, and those with centers at infinity. The former variety includes all of the usual type of circle, the latter variety having its radii in each case a system of parallel lines consists of straight lines perpendicular to the parallels. By this definition a straight line is shown to be a special case of a circle.

By using in the geometry of the hyperbolic plane this definition of a circle as the orthogonal trajectory of a pencil of lines there are found to be three varieties of circles according as the vertex of the pencil is in the finite, infinite, or ideal part of the plane. When the vertex or center is an ordinary point in the finite regions of the plane the circle resembles Euclid's circle in appearance. When the center is at infinity the circle receives the special name oricycle. When the center is an ideal point the circle is everywhere equally distant from the common perpendicular or axis of the pencil, and it is called an equidistant curve. This latter curve has two branches symmetrically arranged around the axis of the pencil. Two important properties of the oricycle may be given. The first is that all oricycles are congruent, the second is that the oricycle furnishes a natural unit of length in hyperbolic space, for it can be proved that the length of an arc of an oricycle that is contained between a point P and a point Q at which the tangent to the curve is parallel to the radius through P is constant in length. This constant has different numerical values according to the unit of length adopted, but it may itself be used as a unit. This length is called the space constant, and is evidently a positive quantity.

Upon extending these conceptions to space of three dimensions the three types of circles are found to have their counterparts in three types of spheres. The sphere with center at infinity is called the orisphere. Figures on this surface with sides formed

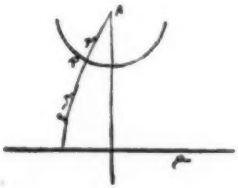
by arcs of oricycles come under the laws of a geometry that is Euclidean, for the sum of the angles of a triangle on this surface is a straight angle.

Leaving this hyperbolic space we come to the second of the non-Euclidean geometries, the one that has received the name elliptic geometry. In this elliptic space no line can be drawn through a point parallel to a fixed line, all lines are finite in length, and the space itself having no infinite region is finite though unbounded.

In an elliptic plane take a fixed line a, and erect perpendiculars to it. Since no lines are parallel these perpendiculars must meet each other, and it is easily shown by the properties of isosceles triangles that they all meet in a common point A, and that this point is at a fixed distance from a. This fixed distance is a linear constant that forms an absolute unit of length for elliptic space. It is usually said to have the length of a quadrant. The point A is said to be the absolute pole of the line a, while the line a is said to be the absolute polar of the point A. By symmetry the perpendiculars when extended through a meet in a point A' symmetrically placed with respect to a. If A' is distinct from A the geometry of this space is strictly analogous to Euclidean spherical geometry, and is therefore of no special interest. The length of a line is measured by four quadrants. If A' coincides with A then the geometry of the space is a new geometry, and the name elliptic geometry is reserved for this case. In elliptic geometry the length of a line is two quadrants.

In this elliptic plane the sum of the angles of a triangle is greater than a straight angle. The plane differs from the plane of the Euclidean and hyperbolic spaces in possessing the properties of a one-sided surface, that is, a straight line cannot divide it into two parts, and consequently a point can move in the plane from one side to the other of a straight line without crossing the line.

Since the elliptic plane contains only one kind of point it contains only one variety of circle, but since the distance from a point to its absolute polar is always q, a quadrant's length, this circle may also be considered an equidistant curve with respect to the absolute polar of its center. Any point P on the circle with radius r is at a distance $q - r$ from the absolute polar of the center.



In elliptic space of three dimensions any two lines not in the same plane have two common perpendiculars, one giving the shortest distance between the lines, the other the greatest distance. In this three dimensional space it is also possible to find straight lines that are everywhere equidistant. Since in Euclidean geometry the equidistance of a pair of parallel lines is one of their most important properties, and since we have no proper parallels in elliptic space, it has been proposed to call these the parallel lines of elliptic space.

In elliptic space there has been found a surface known as Clifford's surface. The geometry of figures formed upon this surface by geodesics or shortest lines conforms to the Euclidean laws. That is, in both elliptic and hyperbolic space surfaces are found for which the geometry is that of Euclid. In this way the consistency of these non-Euclidean geometries is shown to be connected with the consistency of Euclidean geometry. Moreover both hyperbolic and elliptic geometry may be represented by the geometry of figures upon certain surfaces in Euclidean space, a process that again connects the validity of the theorems of the non-Euclidean spaces with the validity of theorems of Euclidean space.

In each of the types of geometry so far considered—Euclidean, hyperbolic, elliptic—there appears in the formulas of their trigonometry and analytical geometry a constant that differentiates the space in which it appears from the others. This "space constant" is imaginary for elliptic space, positive for

hyperbolic space, and infinite for Euclidean space. Various tests have been suggested for discovering the space constant of the space we live in. Most of the methods depend upon measurements of some kind, and so are subject to error. Moreover most of the tests suggested depend themselves implicitly if not explicitly upon the assumption that the geometry of our world is Euclidean. There is small wonder that at present we can only say that some measurements indicate that our space constant is very large and positive, while other theories indicate that we live in an elliptic space. So far there is no rigorous test.

In Cayley's notion of the absolute is found a fairly satisfactory way of picturing the relations between the three geometries. Confining ourselves to space of two dimensions we say that in each plane of a given space there is a conic called the absolute. Two lines intersecting inside this conic are said to intersect in ordinary points, two lines intersecting on the conic are said to intersect at infinity or to be parallel, two lines that meet outside the conic are said to be divergent or non-intersecting lines. Points inside the conic are ordinary points, on the conic are points at infinity, outside the conic are ideal points. In hyperbolic planes the conic is real, and the three types of points exist; in elliptic planes the conic is imaginary, and only the ordinary points are present; in Euclidean planes the conic is degenerate, consisting of consecutive straight lines, and the two types of point present are ordinary points and, on each line, two consecutive points at infinity.

So far our assumption has been that these three geometries characterize three different spaces. It is only just however to quote Poincaré in his contention that these three geometries are merely three different methods of treating one and the same space. He says that it is as foolish to ask which is the geometry belonging to the space of our experience as to ask whether the yard or the meter is the correct unit of length. One uses the measure that is most convenient for one's purpose, and since the Euclidean hypothesis is by far the simplest and easiest to handle it is the part of common sense to use it in any space discussion.

Other spaces beside these three have been suggested and their properties investigated, but the prime importance of the three

lies in the fact that for space of three dimensions and over these three are the only ones that allow of free motion of a body. In two dimensional geometry there are other spaces that permit of free motion, but this limitation of dimensions makes them less important for us,

In the three geometries just considered theorems about perpendiculars are the same. In a plane only one line may be drawn perpendicular to a line at a given point. In space of three dimensions only one line may be drawn perpendicular to each of two perpendicular lines at their point of intersection. That is, in three dimensional space it is as impossible to draw through a point four lines that shall be perpendicular to each other by pairs as it is to draw in a plane three lines that are perpendicular in pairs. If however there were a fourth dimension it would possess this very property of permitting four mutually perpendicular lines to be drawn through a point, and it is a very interesting thing to assume this four dimensional space, and study the geometric configurations that exist in it. Each one of the non-Euclidean geometries may be extended into four-dimensional space, but the fourth dimension of Euclidean geometry is the only one to be dealt with here.

The space of four dimensions that is now to be considered is called a hyperspace. In this hyperspace spaces of three dimensions exist as planes exist in three-dimensional space or lines in two-dimensional space. These three-dimensional spaces are called hyperplanes, and the space of our experience is merely one of these hyperplanes.

In the study of solid geometry one of the first things to do is to find the various ways in which a plane may be determined, since all the theorems of plane geometry may be applied to any parts of the solid figure under discussion that lie in a plane. In the same way, for work in hyperspace, the first thing to do is to determine what groupings of elements determine a hyperplane, in order to make use of the theorems of solid geometry.

A hyperplane is determined by four points not in the same plane, or by a plane and a point outside the plane, or by a plane and a line that intersects the plane but does not lie in the plane, or by three lines through one point but not lying in one plane, or by two planes intersecting in a line.

Some of the direct conclusions from these statements are the following:

A line and a plane can never meet unless they exist in the same hyperplane;

Two planes not in the same hyperplane cannot have more than one point in common;

Two hyperplanes that intersect each other have a plane in common.

A straight line divides a plane into two distinct parts in such a way that passage from a point on one side of the line to a point on the other side necessitates crossing the line. The line does not divide space of three dimensions into two parts, since it is easy to see that by using the third dimension a point may pass completely around the line keeping as close as possible to any point on it, and yet need not cut the line. Similarly a plane divides a hyperplane into two distinct parts. Motion from a point on one side of the plane to a point on the other side implies passage through the plane. But this plane does not divide hyperspace, so in space of four dimensions it is possible to go completely around the plane from one side to the other, keeping as close as possible to any point on the plane, and yet not to cut the plane.

A plane in hyperspace has an indefinite number of perpendiculars drawn to it at any point in the plane. Each perpendicular, together with the plane, determines a hyperplane, so that for three-dimensional geometry only one perpendicular may be drawn to a plane at a point in the plane.

A line in hyperspace is perpendicular to a hyperplane at a point P when it is perpendicular to every line in the hyperplane that passes through P. Such a perpendicular cannot be visualized by our minds that are used to thinking in terms of three dimensions.

When the perpendicularity of planes is considered there prove to be two varieties. The first is familiar from solid geometry. Two planes are said to be perpendicular to each other when they lie in the same hyperplane, and form there a right dihedral angle. Each plane contains one line of the other, and lines may be drawn in each perpendicular to the other. The second variety is called absolute perpendicularity. Two planes that have only one point in common and so lie in two different hyper-

planes are said to be absolutely perpendicular when every line in one through the common point is perpendicular to every line in the other through that point.

In the Euclidean hyperspace under discussion here there are some interesting theorems about parallels, such as the following:

If a line is parallel to a hyperplane all points of the line are equidistant from the hyperplane;

If a plane is parallel to a hyperplane all points of the plane are equidistant from the hyperplane;

If two hyperplanes are parallel, they are everywhere equidistant.

Finally let us consider the three configurations in hyperspace that correspond to three simple solids in a hyperplane. These figures are called hypersurfaces, and are analogous either in construction or definition to the three solids, the cube, the sphere, and the tetrahedron. To construct a cube consider a square moved perpendicular to its plane through a distance equal to a side. The four sides of the square each generate a face of the cube, and the remaining faces are formed by squares in the original and final positions. Now let the cube move out into hyperspace in a direction perpendicular to the hyperplane of the cube and through a distance equal to an edge of the cube. The figure so formed is called a hypercube, and is bounded by eight cubes that are called its cells.

A hypersphere is the locus in four dimensions of points equally distant from a given point. By permitting the hypersphere to pass through our three-dimensional space we get consecutive three dimensional sections, that give some idea of the nature of the hypersphere. These sections are all spheres varying continuously from a point sphere through spheres that increase in size until a maximum is reached in the sphere through the center of the hypersphere and then decreasing to a point sphere once more. This hypersphere has great spheres and small spheres analogous to the great circle and small circles of a sphere, and there is a geometry on a hypersphere that is quite independent of the space in which it lies, as spherical geometry is independent of the three-dimensional space in which the sphere exists.

The tetrahedron may be constructed most easily by constructing on the sides of an equilateral triangle three other equilateral triangles, then rotating the last three from the plane into three-

dimensional space until the free edges and vertices come together, when a solid figure has been formed that incloses a portion of three-dimensional space. It is important to notice that every point in each face of the tetrahedron is as close to the three-dimensional space inclosed by the surface as any other point. Any point in a face moves into the three-dimensional space from where it is without approaching the edge of the face. The figure in hyperspace that corresponds to a tetrahedron in a hyperplane is a pentahedroid. To construct it four regular tetrahedra are formed on the four faces of a given tetrahedron. The last is kept fixed, while the four are folded up into hyperspace until they fit together and inclose a portion of hyperspace. Each point in these bounding tetrahedra is as close to the included four-dimensional space as any other. A point passes immediately from a position anywhere in the tetrahedron to a point in the interior of the pentahedroid without moving through the tetrahedron toward its faces or edges.

Other configurations in the hyperspace of four dimensions have been studied, and other spaces of five, six, . . . n dimensions have been used for various purposes.

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CAUSES OF FAILURE IN PLANE GEOMETRY AS RELATED TO MENTAL ABILITY [1]

By LILIAN L. CRAFTS
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At the beginning of the second term in February, 1922, in the East High School, Rochester, N. Y., a study was made to find out to what extent failure in the first term's work in geometry was due to lack of mental ability and to what extent it was due to other causes.

Two groups of pupils were considered in this investigation. The first group consisted of sixty-two pupils who had failed in their first term's work in geometry. They were assigned to two special classes to repeat the subject. The second group was made up of forty-five pupils selected from three classes who were to go on with the second term's work. These forty-five pupils had passed the first term's work after taking it only once. In these classes those repeating the second term's work and those who had passed the first term's work after a second attempt were excluded. The pupils chosen were really representative. The first group included sixty-two of a possible eighty-six, the second group forty-five of a possible two hundred and twenty-six, some of whom would have been ineligible. The make-up of classes is usually determined by the requirements of the schedule, not with reference to intelligence. This second group, therefore, though small in number, would probably meet the requirements of a random selection.

"The Terman Group Test of Mental Ability" was given to both groups of pupils, and the results were compared with standards furnished by Dr. Lewis M. Terman of Stanford University. These tests have been given to over 30,000 school children throughout the United States, and the results tabulated for age and grade. The median score for 4800 pupils in the second year is 122 points of a possible 220. Our failure group (of second year pupils) had about the same median, or to be exact, 121½. In terms of age, 120 is the score to be expected for pupils aged fifteen years and three months. Our pupils who passed, had a median score of 158, which is ten points better than the median for 3500 pupils of the fourth year of high school. In terms of age, 158 corresponds to seventeen years and two months. It is worthy of consideration that the median score which our failing pupils attained was the score to be expected from pupils

[1] The material on which this report is based was prepared for a paper read before The Mathematics Club of Rochester, N. Y., in February, 1923.

of their age and grade, that is from second year high school pupils, while those passing have the score to be expected from seniors. In other words, does it require the ability of a fourth year student to pursue successfully second year work? The distribution of cases is given in the accompanying tables.

Table 1, Classes 1 and 2

Failures in Geometry Term

Terman Group Tests of Mental Ability

Classes 1 and 2. East High School
Pupils Who Failed Rochester, N. Y. Date of Exam.
Geometry I-1. 1-24-22.

	Class 1			Class 2	
Pupil No.	Age Yr.Mo.	Score	Pupil No.	Age Yr.Mo.	Score
1	15-10	173	1	17-9	171
2	19-4	163	2	18-1	166
3	15-10	158	3	17-4	156
4	15-6	145	4	16-7	155
5	17-6	143	5	14-5	153
6	16-10	142	6	17-0	152
7	16-0	140	7	15-8	151
8	17-3	133	8	15-8	149
9	15-9	133	9	15-6	145
10	15-5	132	10	17-3	145
11	15-9	125	11	14-9	145
12	18-9	123	12	17-3	144
13	15-7	121	13	16-6	142
14	16-3	120	14	16-6	141
15	17-6	117	15	15-7	128
16	15-3	116	16	17-2	126
17	16-9	113	17	15-2	126
18	16-3	113	18	15-9	122
19	15-8	111	19	14-8	122
20	-5	111	20	15-	115
21	15-1	109	21	16-9	113
22	15-2	102	22	15-5	112
23	16-9	100	23	14-4	100
24	14-0	94	24		95
25	15-5	89	25	17-7	92
26	14-8	88	26	16-1	92
27	15-11	86	27	16-7	91
28	15-2	80	28	16-9	88
29	14-10	76	29	16-8	86
30	16-8	65	30	16-	74
31	16-3	50	31	16-1	48

Table 2, Classes 3, 4, 5

Promotions from Geometry (Without Previous Failures)

Terman Group Intelligence Test

East High School

Classes 3, 4 and 5.
Pupils Who Passed
Geometry I-1.

Date of Exam.
1-30-22.

(Class 1)			(Class 2)			(Class 3)		
Pupil No.	Age Yr.Mo.	Score	Pupil No.	Age Yr.Mo.	Score	Pupil No.	Age Yr.Mo.	Score
1	16-3	168	1	15-0	199	1	14-4	197
2	15-0	167	2	15-2	187	2	14-9	182
3	15-2	166	3	17-0	185	3	14-4	178
4	15-8	160	4	16-2	182	4	15-7	177
5	16-7	158	5	15-10	171	5	15-8	171
6	15-6	157	6		170	6	13-10	168
7	14-9	154	7	14-10	160	7	18-5	167
8	16-1	147	8	15-0	159	8	15-6	161
9	14-9	146	9	16-6	140	9	15-5	160
10	14-9	131	10	16-2	125	10	16-2	159
11	15-7	128	11	15-10	118	11	16-10	152
12	15-5	118	12	17-9	116	12	14-10	127
13	15-7	111	13	15-2	114	13	15-8	122
14	15-10	111				14	14-11	115
15	15-4	90				15	16-10	111
						16		92
						17	17-2	81

Table 3

Grade Standards for Terman Group Test

Percents equalling or or exceeding	Percentile scores by grade 7	8	9	10	11	12
1 percent equal or exceed	147	171	186	194	200	205
2½ " " " "	134	158	173	185	192	199
5 " " " "	123	147	164	179	186	193
10 " " " "	109	134	152	167	177	185
15 " " " "	100	125	143	159	171	179
20 " " " "	93	118	136	152	164	174
25 " " " "	88	112	130	147	158	169
30 " " " "	83	106	124	141	153	165
40 " " " "	75	97	114	131	143	157
50 " " " "	67	87	105	122	134	148

Grade Standards for Terman Group Test (Continued)

60	"	"	"	"	60	79	96	113	125	139
70	"	"	"	"	53	71	87	103	116	129
75	"	"	"	"	50	66	83	99	111	123
80	"	"	"	"	46	62	78	90	105	117
85	"	"	"	"	42	56	72	86	100	110
90	"	"	"	"	37	51	66	79	90	101
95	"	"	"	"	31	42	54	67	78	86
97½	"	"	"	"	25	33	45	57	68	74
99	"	"	"	"	20	29	35	47	57	62
Number of cases for each grade					5582	6928	7648	4889	2194	3416

Total Number, 30,657.

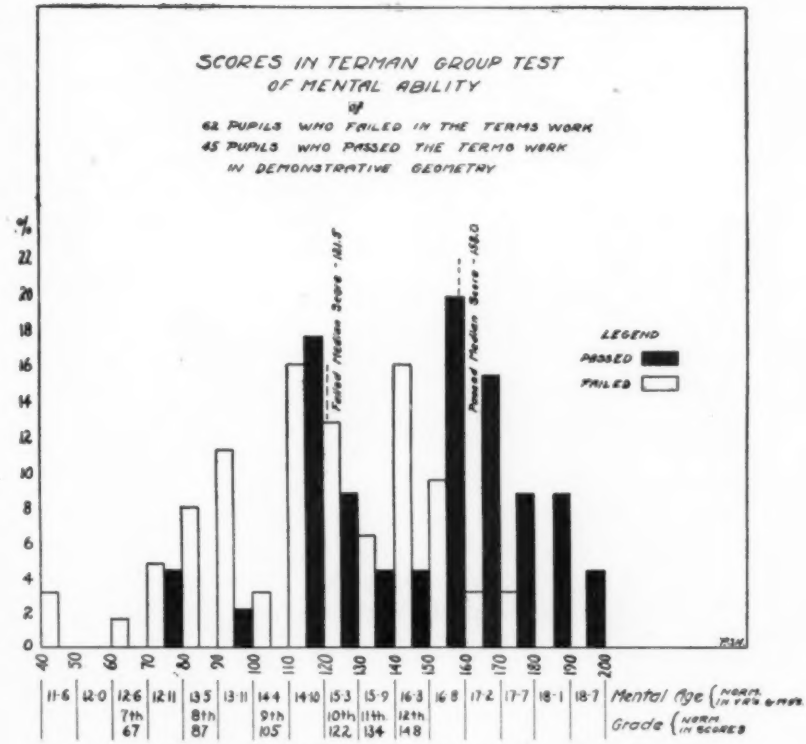
(The table above is furnished by Prof. L. M. Terman of Stanford University)

Table 4

Mental Age Equivalents for Terman Group Test Scores

Score T.G.T.	Equiv. M.A.	Score T.G.T.	Equiv. M.A.
220	20-0	110	14-10
215	19-9	105	14-7
210	19-6	100	14-4
205	19-3	95	14-1
200	19-0	90	13-11
195	18-9	85	13-8
190	18-7	80	13-5
185	18-4	75	13-2
180	18-1	70	12-11
175	17-10	65	12-9
170	17-7	60	12-6
165	17-5	55	12-3
160	17-2	50	12-0
155	16-11	45	11-9
150	16-8	40	11-6
145	16-5	35	11-4
140	16-3	30	11-1
135	16-0	25	10-10
130	15-9	20	10-7
125	15-6	15	10-4
120	15-3	10	10-2
115	15-1	5	9-11
		0	9-8

(Tables 3 and 4 are given by the courtesy of Prof. L. M. Terman of Stanford University, California.)



GRAPH SHOWING DISTRIBUTION OF CASES

Passed Geometry Term			Failed Geometry Term		
No. Pupils	% based on 45 cases	Score in Intelligence Test	No. Pupils	% based on 62 cases	Score in Intelligence Test
0	0	41-50	2	3.2	41-50
0	0	51-60	0	0	51-60
0	0	61-70	1	1.6	61-70
2	4.4	71-80	3	4.8	71-80
0	0	81-90	5	8.0	81-90
1	2.2	91-100	7	11.2	91-100
0	0	101-110	2	3.2	101-110
8	17.6	111-120	10	16.0	111-120
4	8.8	121-130	8	12.8	121-130
2	4.4	131-140	4	6.4	131-140
2	4.4	141-150	10	16.0	141-150
9	19.8	151-160	6	9.6	151-160
7	15.4	161-170	2	3.2	161-170
4	8.8	171-180	2	3.2	171-180
4	8.8	181-190	0	0	181-190
2	4.4	191-200	0	0	191-200
Median 158 from actual scores 154 from grouped scores			Median 121½ from actual scores 122½ from grouped scores		

The work of the second year classes is not designed to cut off all pupils below the middle rank. Hence, including all pupils who were twenty points or less below the median in a group of whom success might have been expected, we might conclude that 30% of the failures were due to lack of the necessary mental ability. Moreover, there is so little overlapping of those passing in the scores below 100, that it seems as if a score of 100 might be considered as a minimum requirement. About 30% of a second year class would naturally fall below this score. The final semester record, showing 86 failing, 226 passing and 42 who left school or dropped the subject, gives about the same result, namely 28% of failures. This includes those failing after a second attempt, while the two groups tested include no such pupils, as they are not permitted to take a subject a third time, unless certain conditions have been met.

The astonishing thing is that those who pass the first semester in geometry rank so high: only 6% were below 100; 13% were between 110 and 120; while the difference between the medians of the two groups mentioned above is 36 points.

The amount of overlapping near the median and the large number of failures of pupils above the median would lead to the conclusion that other causes of failure were operating with the pupils who ranked above 110. The median for the combined groups might be of interest. It is 140, but it is of no significance because the combined group would not be a normal group.

For purposes of comparison, reference may be made at this point to a report on a failure class in algebra. This report was published by Mr. O. A. Wood of Kansas City, Mo, in the "Twenty-first Year Book of the National Association for the Study of Education." The Stanford Revision of the Binet-Simon Tests and the Rugg and Clark Algebra Tests were given to a first-year algebra class. The coefficient of correlation (by the Spearman foot-rule) between intelligence quotients and class grades was .993. Mr. Wood concludes: "Since there is a close relation between general intelligence and ability to learn algebra, it seems reasonable to conclude that the general intelligence of each pupil should be determined before he is required to take the subject. If he is clearly below normal, in general intelligence, he should be prohibited from taking algebra, unless there

are good reasons to the contrary." Many a teacher of algebra would echo Mr. Wood's recommendation for those "clearly below normal."

In the case of the geometry groups mentioned above, a questionnaire was distributed asking information as to the home conditions and outside activities of the pupils. This was done in order to find out whether there were external causes of failure. The results from this questionnaire were somewhat meagre as only 35 of the failure group complied with the request. The report is summarized below:

No. Pupils	Nationality of Parents
23	American
1	English
1	Scotch
1	Italian
1	Polish
2	German
6	Jewish

It may be concluded that the first 25 pupils mentioned in the table given above, or about 70% of those replying, had no handicap due to lack of knowledge of the English language. Hence, about 30% may have failed for linguistic reason. But the same difficulty would have caused a low score in the intelligence test. By comparing the records of individuals, we find this to be the case. Eliminating several who have a low score from the number of failures to be attributed to lack of knowledge of English, probably only 10% of the failures of pupils above the median can be ascribed to this cause.

The home conditions of the pupils as judged by the occupations of the parents would indicate no economic handicap. The parents were skilled laborers, business men of more than average rank, as managers, and manufacturers, and professional men. In only two cases were the mothers working. One of these was a clerk, the father being dead, and in the other case, both parents were teachers.

The attendance records for the entire number of failures show that three failed because of illness and two because of change of schools.

The number of subjects taken varied as given in the following table. The usual program for all pupils is four recitations, with

permission to take an additional period of work for which no outside study is required. Gymnasium classes are also additional and bring the program up to five or six periods of work in some instances.

No. Pupils	No. Recitations
24	4
8	5
2	5 on 2 days
	6 on 3 days
3	6

Inasmuch as the results in the intelligence test show that only the last five pupils in this table were in the lower rank in mental ability, we cannot ascribe the failures of those above the median to this cause.

The pupils were asked to state how many study periods in school and how many minutes each day were spent in the preparation of lessons. While we cannot rely on the statements as to time actually spent, it may be assumed that this amount of time was available for study.

No. Pupils	No. Hours of Study
18	3 or more
13	2—3
5	1—2

Of the last five pupils, two were above the median and their failure might be ascribed to lack of study, while in the case of the other three, it was doubtless a contributing cause.

In regard to the outside activities of the pupils, the following information is given.

No. Pupils	No. times per week attended theatre
5	2
16	1
5	seldom
2	0

In the school activities, athletics, school paper, etc., 15 took part, out of 35 reporting. Membership in church clubs was claimed by 8 of the 35.

The forms of recreation which appealed to these students were various forms of athletics, music, photography and reading.

No. Pupils	Recreation
25	Baseball, tennis, riding, etc.
4	Band, orchestra, piano-playing.
1	Photography.
1	Reading.

One pupil listed attendance at Hebrew school for ten hours a week as recreation. This pupil was one of the cases where failure in school had followed a long illness. Two pupils were giving ten and twelve hours, respectively, to piano practice. Their scores in the intelligence test were 100 and 149, respectively. Both pupils failed the second time. Music may have been one thing too many.

In answer to the question, whether they planned to go to college, the replies were:

No. Pupils	Reply
18	Yes.
9	No.
3	Undecided.

The number of other failures in high school subjects is to be considered in connection with the fact that these pupils were in the second term of the second year. They had had twelve possible opportunities to succeed or fail.

No. Pupils	No. of Other Failures
16	0
13	1
4	2
3	3
1	4

The pupils explained their own failures in geometry as follows:

No. Pupils	Reason for Failure
14	Difficulty of the subject.
10	Lack of explanation.
12	Lack of study.
7	Irregular attendance.
1	Change of teacher.
1	Change of school.

The teachers in their term reports explained these failures in geometry somewhat similarly:

No. Pupils	Reason for Failure
29	Inability.
4	Carelessness.
22	Lack of study.
3	Irregular attendance.
3	Change of schools.

The pupils considered in the tabulations above are only those enrolled in the group of failures. After considering the data for the different individuals, the following concluson seems reasonable.

Percent of Cases	Reason for Failure
30	Mental incapacity.
10	Difficulty in the use of English.
4	Irregular attendance.
3	Change of schools.
47% Total	Beyond the control of the schools.

The 53% remaining constitute a challenge to the efforts of parents, administrative officers and teachers.

The question "What can be done to remedy this situation?" at once arises. The recent report by "The National Committee on Mathematical Requirements" on "The Reorganization of Mathematics in Secondary Education" covers this field as far as experimentation has gone. Anyone who attempts any simple investigation is convinced of the need of the further use of tests of general mental ability and also tests of mathematical ability, such as the Rogers Tests. Classes made up of pupils able to do the work would then result.

Some progress in reducing the number of failures might be made if a campaign with that end in view were begun and carried on for several years and the records published from time to time to show progress. It would be necessary to use some standardized tests in this connection also, in order to be sure that the gain was a real one and not attained by lowering the standard of promotion.

The following items might be part of this campaign:

1. Give an intelligence test to every pupil before he is assigned to classes. We now have this record in Rochester for all pupils entering from the grammar schools. It would be necessary to give the test to the others, entering from the parochial schools and out-of-town schools. These results could be translated into an intelligence quotient and entered on the permanent record cards. Multiplication of the actual age by the intelligence quotient would then give the approximate mental age at any term when question of ability arose.

2. On a set of cards, have the pupils record name, age, nationality of parents, previous success in mathematics, reason for any previous failure, and plans for the future. This information is most useful in the hands of the classroom teacher. In large schools, information of this character is sometimes a matter of record but many teachers are not aware that it is available.

3. Eliminate from the classes those who seem incapable of the work. The present study would suggest that pupils whose mental age was below fourteen years might profitably defer demonstrative geometry or choose a course in which it is not a required subject.

4. Stimulate the pupils to greater effort by the use of immediate incentives, such as exhibits of good work, honorable mention, publication of honor rolls and such modification of classroom procedure as study and experimentation may suggest.

In conclusion, it seems that 50% of the failures are probably preventable. Further experimentation is necessary to devise means to find out which pupils are incapable. By concentrating efforts on those who are able to do the work, by greater stimulation, and by better teaching, the standard of all the pupils might be improved.

The statements made in the report given are to be considered as tentative only. In many respects, the work of the investigation was incomplete and the number of pupils considered was small. The Terman Group Test has been used in other schools and correlations determined between scores in the intelligence test and teachers' ratings in high school subjects. The correlations usually are not high, but high correlations are not to be expected. In one such investigation in the high school of Palo

Alto, California, the writer followed up all the cases in the freshman class where there was a marked discrepancy between the teachers' ratings and the rank in the intelligence test. There were 19 such cases out of 148 considered. The Binet-Simon test was given to each of the 19 pupils and inquiry was made in regard to home conditions, habits of study, etc. In 17 cases, adequate explanation of the discrepancy was found. Pupils having a high score in the intelligence test and a low school record were found to be contributing toward the support of the family or to be handicapped in various ways. Pupils having a low intelligence score and a higher rank in their school record were found to be spending long hours in study or to have very attractive personal qualities. The writer feels, therefore, that a group test of intelligence is a very useful aid to good judgment in dealing with groups or with individuals.

It seems desirable, in making an investigation in regard to any one subject to tabulate separately, the results for those repeating the subject. The relation between intelligence scores and success or failure in school is obscured when results for those taking the subject a second time are included in the same tables with results for those taking the subject for the first time.

This paper is offered in the hope that it may be of some service to those who do the daily work of the classroom and who may be contemplating some such study with their own pupils.

NOTE.—The intelligence quotient is the result obtained by dividing the mental age as determined by an intelligence test, by the chronological age. Thus a pupil 8 years old mentally but 10 years old chronologically has an intelligence quotient of 8/10 or .80, usually read as 80, ignoring the decimal point.

GEOMETRY AS A COURSE IN REASONING

By HENRY P. McLAUGHLIN
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How can geometry be reclaimed from its present position in most of our secondary schools as little more than an unsuccessful course in formal mathematical logic, is a question that has engaged the attention of alert teachers for many years. There is a movement on foot at the present time to overhaul the whole of our mathematical teaching in all schools below the college grade which is doing much to improve the situation. As soon as school authorities throughout the country had decided that geometry could and should be taught in the newly organized junior high schools, the problem was put squarely up to mathematics teachers to reorganize the content and methods of this subject. It was quite evident from the start that the old style presentation of mathematical proofs that we call demonstrative geometry would not be understood by pupils below the ninth grade, and there was a sneaking suspicion in the minds of many that it was not too well understood by most pupils above that grade. Accordingly Euclid's organization of the subject matter of geometry was thrown boldly overboard by mathematical committees appointed to study the situation, and a course of study was recommended which was based on the capacities of childrens' minds rather than on those of the ancient Greek philosophers. Rigid proofs were eliminated and intuitive, observational, inventional, and numerical geometry was substituted.

The question naturally suggests itself as to whether junior high school methods of teaching geometry can be successfully used in the upper grades and to what extent. Shall the rigid proof be abandoned in the senior high school also where it often seems to be over the heads of the pupils? Most mathematics teachers would say no. All of us will agree that the kind of demonstrative geometry which attempts to prove theorems as self-evident as some of the axioms themselves, must go; that very subtle forms of proof should be eliminated; and that material similar to that prepared for the junior high school may be profitably used in the tenth grade; but most of us also feel that

mathematics should be something more than a study of the properties and relations of quantity and space. It matters but little to the average mathematics teacher how often the statement is made that the theory of formal discipline has been exploded. He has an unshakable faith that the mind *can* be disciplined through a study of mathematics. The general public is apt to think of him as a rather cold and clammy animal but a study of literature on mathematics teaching shows him to be an ardent enthusiast concerning his own subject. The following quotation from an article by a professor of mathematics is a good illustration. "And now what shall we say is the ideal of excellence that hovers above thought activity? What is the angel that woos our loyalty to what is best in that? What is the muse of life in the great art of thinking? An austere goddess, high, pure, serene, cold towards human frailty, demanding perfect precision of ideas, perfect clarity of expression, and perfect allegiance to the eternal laws of thought. In mathematics the name of the muse is familiar: it is Rigor—Logical Rigor—which signifies a kind of silent music, the still harmony of ideas, the intellect's dream of logical perfection."

The chief trouble, however, with such teachers in the classroom is that while hovering with the angels, they soar high above the heads of their pupils. Certainly this beautiful ideal of logical rigor is attained by few of them. It is true that many learn to reason rigorously while dealing with mathematical concepts but most of them fail miserably to transfer this power in dealing with problems outside the field of mathematics. Psychologists in recent years have shown that if the pupils are to make such a transfer, most of them must be taught how to do so, as it is not made automatically by some sub-conscious cerebral mechanism. Consequently if demonstrative geometry is to be made, a training in reasoning which may be used in other fields, there must be radical changes in the methods of teaching it.

It is impossible to make an exhaustive survey within the limits of this article of the various ways in which our pupils will be called upon to use their reasoning ability, when they leave school. Only a few of their most pressing needs will be touched upon to illustrate how a transfer of training may be made. All of them will be called upon to perform the duties of citizenship. All of them will have to solve what we call for

lack of a better name problems of everyday life; for example: how to run a furnace economically, how to keep small boys from robbing our pear tree, etc. It is admittedly the special province of the teachers of the social sciences to train for good citizenship but invariably they find themselves handicapped by the failure of most of their pupils to decide whether an argument is logical or not. He looks very often to the English teacher for assistance as he knows that some space is given to argumentation in nearly every textbook on English composition and as a rule he does not look in vain. Much less often he expects help from the mathematics teacher. This should not be the case.

Lack of skill in examining an argument is responsible for what is perhaps the greatest failure that our secondary school graduates make as citizens. They do not vote as intelligently as we have a right to expect. It is true that it is not an easy matter always even for an expert to pick the best candidate. Most of us have not the time to make a detailed study of the record of a candidate from original sources or to make a first hand examination of the conditions dealt with in the platform on which he stands. As a rule we must base our judgments on political speeches, newspaper and magazine articles, and discussions. All of the evidence from these sources is more or less biassed and yet contains much truth. The question for us to settle is how can we train our pupils to protect themselves from fallacious reasoning and to sift out the true from the false. Textbooks on argumentation usually treat logic as being primarily an offensive weapon. The geometry teacher has a splendid opportunity for showing how it may be used on the defensive.

The kinds of argument that may mislead are numerous but only those which the teacher may train his pupils to guard against without getting too far away from geometry, will be mentioned here. One of the commonest sources of error is the vagueness with which the plank of a platform is stated and often intentionally so by the politicians who framed it. The teacher should insist on a clear cut statement of the theorems studied in class though not necessarily in the exact words of the textbook. In fact with very little training, a class will be able to pick out the few theorems in the book, which are not stated with sufficient clearness; and with a little help will restate them

correctly. The next step is to contrast the precision of the typical mathematical statement with vague statements of political issues selected from the platforms of all parties.

A closely allied group of fallacies springs from faulty definition or no definition at all. This is almost universal in political speeches and discussions. In a recent campaign, many voters argued for a vicious candidate as follows: "Well, what if he has been convicted of being crooked. All politicians are crooks and we had better vote for the one who has been friendly to us personally rather than for a crook who has never done anything for us." Evidently the fallacy arose from a lack of definition of the word crook. Further discussion brought out the fact that they put the man whose worst crime was using a street car transfer after the time limit had expired in the same class with the man who would sell his vote to the highest bidder if elected. Their argument then implied that since all the candidates might thus be incorrectly squeezed into the same class of undesirable citizens, they must be equally undesirable. This error is by no means uncommon even among educated people. A comparison of careless usage of this kind with the precise definitions of geometry is enough to show why rigid proofs are the rule in this subject and the exception in political discussions.

Begging the question or arguing in a circle is a fallacy that crops out quite often even in geometry classes. Such an error by some pupil presents a good opportunity for showing why it is wrong and how common a mistake it is. Take for example the theorem that the angles opposite the equal sides of an isosceles triangle are equal. When this is assigned as an original, occasionally the following proof is offered. A construction line is drawn to bisect the angle at the vertex and the pupil proceeds to prove the theorem by congruent triangles. The theorem he remembers best in this connection is the one about the two angles and the included side and he tries to work it into his proof. He points out quite correctly that the angles at the vertex are equal, and also the sides of the isosceles triangle. He is puzzled for a minute to find the second pair of angles but he finally decides that those at the base are equal. Why? Because the angles opposite the equal sides of an isosceles triangle are equal. This makes the circle complete and provides the occasion for a lesson in reasoning.

Another common fallacy is that of drawing incorrect conclusions from true premises, sometimes called a *non sequitur*. How often we find it necessary to say to a pupil who is reciting: "I don't see how that follows at all." To make the nature of this error clear, it may be helpful to make a short study of syllogisms. The classic example concerning the necessity of Socrates' death should be avoided, as most pupils are sure to regard it as a lot of fuss about a very self-evident fact. A better illustration is the following:

Major premise.	All mushrooms are edible.
Minor Premise.	This is a mushroom.
Conclusion.	Therefore this is edible.

This example would bring home beyond any doubt the necessity of examining each of the premises with care, as a mistake in either one of them might mean serious illness or death. But if the premises are correct, the conclusion stated necessarily follows. However, if we should take for our conclusion: therefore we should eat it, we would have a very common form of non sequitur, only it is usually expressed in this way: What's the use of finding mushrooms if you don't eat them? The non sequiturs that occur in the classroom are even more glaring, but occasionally one is sufficiently elusive to require detailed analysis.

The use of the syllogism is perhaps the only part of formal logic that it will be profitable to touch upon, but this much of it should be well worth while, if we are to make our teaching of geometry a training that will function outside the classroom. Very often in a political campaign, an argument is offered that we feel is false and yet we are puzzled when we try to lay our finger on the fallacy. However, if we reduce the argument to the syllogistic form, the error is strikingly exposed. It is interesting to notice that proof of analogy, which cannot be reduced to a syllogism, is never a rigid proof and seldom proves more than a possibility. The conventional arrangement of a proof of a theorem in geometry is very much like the syllogistic form. Each statement is the conclusion drawn from a major premise which is called "the reason" and a minor premise that is usually understood. Frequently it is brought out by the questions of the teacher.

One other fallacy appears persistently in the classroom and that is hasty generalization. This is partly due to our methods of teaching by types. When a figure is put on the board, some of the class are apt to take for granted that every feature of it is typical. For example, they may get the idea that parallel lines are always horizontal from the conventional figures of the theorems dealing with alternate-interior and corresponding angles. Of course every good teacher guards against this error. It is only one step more to point out how common it is in everyday life. The foregoing list of fallacies is merely suggestive: Most teachers who are interested in this side of geometry will probably wish to add to it.

A reference has already been made to another side and that is to the usefuleness of this subject as a training in the solving of problems. Besides passing on the solutions submitted to us by others, we must often find them ourselves. Here the syllogism is useful only as a sort of acid test for our results. The solving of "originals" forms the best approach to instruction in this kind of reasoning. No matter what the question is that is up before us for settlement, we must first decide what data we have to work on as a basis and just what the special issue is to be dealt with, that is we must first decide what is "given" and then what we have "to prove" or "to find." The methods of actually finding the solutions of originals differ more or less from those employed in the settlement of public questions or problems of everyday life, but all of them have certain very important features in common. In every case it is necessary to marshal all the facts that have any possible bearing on the subject, and to refer to solutions of similar problems. These suggest possibilities that we test in various ways, according to the nature of the problem. We reject those that do not help us and build on those that do. Here persistence is a cardinal virtue. We are all familiar with the pupil who quits easily. He is apt to think that solving an original is like finding Bunker Hill Monument and that all he needs to do, if he is near enough to see it at all, is to look around for a few seconds until he sees it looming up above him. After a few feeble attempts at solving originals, he invariably decides that they belong in the same class with the monuments that are beyond his range of vision. It is extremely important to rub into such pupils the idea that

solving problems of any kind is a trial and error process in which perspiration can almost always make up for shortages in inspiration. The application of these principles to the handling of difficulties met outside the classroom is easily made but the teacher must not fail to make explicit reference to such problems. Some automobile trouble may well be used as an illustration. Nearly every pupil will know of some self-starter which does not live up to its name. They can easily be led to see the importance of calling to mind the mechanical principles involved and the functions of the different parts of the machine on which the proper working of the self-starter depends in order to determine what is wrong with it. The next step is the careful testing of each of these parts to see which one of them is not working properly. Very often a reference of this kind will react on the ability of the mechanically inclined boys and girls and help them to solve their originals. In any case it will be an immense help in bringing out the fact that preliminary study followed by persistent trial is a method of universal application in the solution of problems.

Other ways and other fields might be mentioned in which the reasoning of geometry may be applied, but these examples will perhaps show how the professor's goddess of logical rigor may be brought to earth and made to perform homely tasks with such precision and despatch that our pupils will cease to ask "What good is geometry?"

MATHEMATICS AS USED BY THE ACTUARY

By JOHN M. LAIRD
Connecticut General Life Insurance Company
Hartford, Conn.

Each well established life insurance company has at least one officer known as the Actuary and usually one or more with the title of Assistant Actuary. Under these officers are a number of men who are engaged in actuarial work and who hope eventually to receive official recognition. Most of these men are college graduates with special training in mathematics and with a real liking for mathematical problems.

Every would-be actuary in the United States and Canada hopes to become a Fellow of the Actuarial Society of America with office in New York City or the Institute of Actuaries with office in Chicago. Most of the leading actuaries in these two countries belong to either the Society or the Institute and several are members of both organizations. Fellowship can be obtained by passing six examinations. Sometimes a student passes the first or second examination in his final year of college. More frequently he begins these examinations after graduation and if he passes one each year he is making good progress. This means six years of systematic study after graduation from college.

The first two examinations cover general mathematical subjects—arithmetic, algebra, theory of probabilities, calculus of finite differences and the elements of differential and integral calculus. The third and fourth examinations take up the application of mathematics to life insurance. The last two examinations involve every phase of the life insurance business, including insurance law, selection of risks and investment of funds.

The science of life insurance is based on two types of tables; first, interest and discount tables; second, mortality tables. The American Experience Mortality Table, which is the legal standard in this country, purports to trace the mortality among 100,000 persons entering at age 10. Of these 100,000 persons 749 die during the first year, thus giving a death rate of 7.49 per 1,000. The history is traced from year to year until at age 95 there are only three survivors and as these three die during that year, the table terminates at age 96.

How is such a mortality table formed? Obviously it is impossible to take 100,000 persons at age 10 and record every death each year for 90 years, that is, until the last survivor has died. Furthermore, if such a table could be obtained, it would be out of date by the time it was completed.

In practice, we first find the rate of mortality at a given age by comparing the number of deaths at that age with the number of persons living at that age, that is, with the number of persons "exposed" to risk of death. For instance, if at age 50 there are 1,000 persons exposed during the year and if 14 of these persons die, the death rate at age 50 is 14 per 1,000. When we have thus determined the death rate at the principal ages or at groups of ages, we can by mathematical processes construct a table of mortality.

Such a table in its first form is irregular because of insufficient data or accidental fluctuations. For instance, the crude death rate may be:

Age 50	14
Age 51	16
Age 52	13

It is therefore necessary to iron out these irregularities. This smoothing process known as graduation may be done in a number of ways. Perhaps the graphic method is the simplest, namely, to plot a curve showing the rate of mortality or some other function at each age or each fifth age and then draw a smoother curve conforming as closely as possible with the original. In this work of graduation actuaries frequently use the higher mathematics such as calculus of finite differences, theory of moments and the method of least squares.

The American Experience Mortality Table constructed 55 years ago is still in use as the legal standard. New mortality tables are, however, frequently required to show the modern mortality on standard lives and on various classes of impaired lives.

After the mortality table has been constructed, how is it used? One popular fallacy is that the actuary bases premiums on the expectation of life, that is, on the average number of years of life. For instance, at age 30 the expectation of life according to the American Experience Table is 35 years. This means that

if we have a large number of people at age 30, the average future lifetime of these persons will be 35 years. However interesting this expectation of life may be to the layman, it is of no value in insurance calculations. All our rates are based on the number of people living and the number dying in each year of age and not on the average number of years they will survive. To facilitate these calculations we construct from the mortality table what are known as commutation columns and we resort to various mathematical short cuts. These calculations involve algebraic formulæ and particularly the theory of probabilities.

The premium charged the policy holder is known as the gross premium. It is made up of two parts: first, loading; second, net premium. The loading is supposed to cover expenses. The net premium is based on a certain rate of interest and a definite rate of mortality. If the company actually earns this rate of interest and experiences this rate of mortality, then the net premiums on any block of business accumulated at interest will be just sufficient to pay all obligations as they fall due.

There are two classes of life insurance: first, mutual; second, stock. Under mutual insurance the company assumes a low rate of interest, say, three per cent, a high rate of mortality and a substantial loading. Any saving from higher interest or from favorable mortality or from excess of the loading over actual expenses is either added to surplus or returned to the policy holder in the form of dividends. It is the duty of the actuary to compute these dividends. Under stock insurance the company charges a lower rate based on the probable actual interest earnings, mortality rate and expenses to be incurred during the life of the policy. It is the aim of the actuary to compute these rates in such a way that they will be just sufficient to meet all obligations and to leave the insurance company a small profit commensurate with the services rendered.

What is the legal reserve under an insurance policy? According to the American Experience Table the death rate per 1,000 increases from 8 at age 20 to 14 at age 50 and 144 at age 80. Under an ordinary life policy issued at age 20 the net level premium according to the American Experience Table with three and a half per cent interest is $13.48 per $1,000. In the early years of the policy this premium is more than sufficient to cover the cost of the insurance. The excess is therefore set

aside as a reserve and is accumulated at interest from year to year.

If the insured surrenders his policy after a few years, a substantial part of this reserve may be returned to him as a cash value. One function of the actuary is therefore to compute the reserves and the surrender values.

The actuary does not spend all his time on problems involving the higher mathematics, in fact, only a small part of his work is of a mathematical nature. His mathematical training is, however, of distinct service in analyzing any business problem and in finding a logical and practical solution. Even in the mathematical part of his work it is essential that every answer should be checked up by common sense.

Apparently Americans are by nature or by training more inclined to develop into salesmen or manufacturers than to become actuaries. A large proportion of the actuaries in the United States have therefore come from either Canada or Scotland. Here is an opportunity for the teachers to show an increasing number of pupils that mathematical ability is required in actuarial work and that a thorough knowledge of the science of life insurance frequently helps to solve executive problems in the large insurance companies.

NEW BOOKS

Topics in the Calculus. By GEORGE RUTLEDGE. Ginn & Company, 1923, Pp. 252. A text for an "early" course in the calculus, which may precede analytic geometry, and be taken simultaneously with trigonometry.

Plane Geometry. (Revised Edition.) By JOHN W. YOUNG and ALBERT J. SCHWARTZ. Henry Holt and Company, New York, 1923. Pp. 278.

The typical pupil who begins his demonstrative geometry, usually in the tenth or eleventh school year, has had arithmetic throughout an eight year elementary school, and algebra during the first year, or perhaps the first two years, of high school. The vocabulary of geometry, its symbolism, and the deductive processes with which it deals, are largely unknown to him. (To be sure, this condition will not be true when intuitive geometry shall have been accepted in grades seven, eight and nine). Teachers have found that this pupil needs an introduction to the subject which would prepare for its difficulties of a formal demonstration. Modern texts have, accordingly, provided short introductions of an informal nature, to help the pupil acquire the vocabulary, and the symbolism used in formal demonstrations. Young and Schwartz *feature* this "introduction to the subject" aiming "to provide a rational approach to the subject that will lead the student by easy stages to formal methods." They find "geometric drawing" the natural approach to this end.

By using the following "fundamental propositions" about *motion* the authors have simplified many of the customary proofs.

"I. Any figure in a plane may be imagined to slide along the plane in such a way that every point in the figure moves along a straight line.

II. Any point C of the plane of a figure may be kept fixed and the figure imagined to rotate in the plane about C as a center. Every point of the figure (except C) will then describe a circle with center C.

III. Any two points S and T of the plane of a figure may be kept fixed and the plane imagined to rotate about the line ST as an axis, until it coincides again with its original position, but with its sides reversed.

We shall describe this type of motion by saying that the figure has been turned over about the axis *ST*. During such a motion every point of the axis remains fixed.

Teachers of geometry are giving more attention to helping pupils get an insight into the nature of a proof, and the methods of thinking used in problem solving. That Young and Schwartz give more than the usual emphasis to the former, may be seen from portions of pages 43, 44, 45 and 46:

"The student should now understand, in a general way, the nature of a geometric investigation. During the study of Chapter I he has observed that one object of geometry is to investigate the properties of geometric figures with the view of discovering how to make certain constructions; he has also observed that the method of investigation is a process of reasoning wholly independent of physical measurement. In Chapter II we begin a more systematic study of geometry. It becomes necessary, therefore, to define the following logical terms:

A proposition. A geometric proposition is a statement asserting a property of a geometric figure, or figures.

A geometric proof. To prove a proposition, or a construction, is to show that it follows as a consequence of definitions, propositions, and constructions previously proved or admitted. Such a process is called a demonstration, or a proof.

A fundamental proposition. It is manifestly impossible to prove all propositions. In order to provide a basis for demonstrations, some propositions must be assumed, or admitted, without proof. Such an assumed proposition is called a fundamental proposition, or a postulate. Thus, "One straight line, and only one, can be drawn through two given points" is, in our treatment of geometry, a fundamental proposition.

A theorem. A proposition which may be proved on the basis of the fundamental propositions and definitions is called a theorem. Thus, "the tangent to a circle at a given point is perpendicular to the radius at that point" is a theorem.

A corollary. A theorem which is closely connected with a proposition or a definition and which is easily deduced therefrom is called a corollary.

Thus, "Two straight lines can intersect in no more than one point" is a corollary of the fundamental property of the straight line referred to in 123, and "All radii and all diameters of a circle are respectively equal" is a corollary of the definition of a circle (24, 18, 20).

Qualifications of a definition. Since the definitions are always an important factor in a logical discussion, it is important that we should call attenton to the following properties of any good definition:

(1) A definition should affirm a sufficient number of properties of the thing defined to determine it completely, and no more. (2) The properties affirmed must not contradict previously established propositions.

Examples. "A straight line in the plane of a circle and meeting it in only one point is said to be tangent to the circle" is a correct definition of a tangent to a circle; but "a straight line which meets a circle in only one point and is perpendicular to the radius drawn to the point of contact is called a tangent" is not a correct definition of a "tangent" because it is both insufficient and redundant. It is insufficient because the line and circle must be in the same plane. Why is it redundant?

Undefined terms. It is, of course, impossible to define all terms; a knowledge of some terms must be admitted as a basis for defining other terms. Thus, the terms point, straight line, plane, distance, etc., have been left undefined in this text, a knowledge of the meaning of these terms being assumed.

Hypothesis and Conclusion. A theorem consists of two parts: the hypothesis or that which is assumed or given; and the conclusion, or that which is asserted to follow from the hypothesis.

Thus the theorem cited in 124 may be divided into two parts as follows:

Hypothesis: Given any straight line tangent to a circle at a given point.

Conclusion: The straight line is perpendicular to the radius at the given point.

It should be noted that a formal arrangement of a proof consists of three parts as follows:

Given: Here is stated, with reference to the figure, which accompanies the proof, the hypothesis, or the conditions given in the theorem.

To prove: Here is stated, with reference to the figure, the conclusions asserted in the theorem.

Proof: Here is stated, in concise steps, the argument used to prove the conclusions just stated.

It is important to observe that every assertion in the argument must be based on the hypothesis, a definition, a fundamental proposition, a construction, or a theorem previously proved. In general, the authorities supporting any such assertion should be explicitly stated in connection with the assertion. Thus, in the preceding paragraph (not quoted) the authorities supporting the assertions in steps 2, 3, 4, 5 are written on the right of and opposite to the respective assertions. There are, however, a considerable number of fundamental propositions either so elementary in nature that explicit reference to them each time may generally be omitted in a first course, as "if A, B, C are three points on a straight line, one and only one of the points is

between the other two''; or whose applications are so apparent that explicit reference to them each time would be burdensome, as ''A straight line can be drawn through any two given points,'' ''a circle can be described from any given point as a center and with any given line segment as a radius,'' the fundamental propositions on motion (29-31), of which step 1 in the preceding paragraph is an example, etc.

No statement in a proof is warranted simply because it appears to be true from the figure, no matter how carefully drawn. The fallacies into which one may be led by trusting to appearances may be appreciated from the following examples:'' (Here follow familiar optical illusions.)

The Progress of Arithmetic in the Last Quarter of a Century. By DAVID EUGENE SMITH. Ginn and Company, New York. Pp. 93. 72 cents.

All teachers interested in arithmetic will welcome this little book. It is really an essay of seventy-eight pages, besides the introduction and the publishers' supplement. ''It exploits no special method and no narrow curriculum, but it registers the progress of the past quarter of a century and it records the present status of the oldest and most important of all the sciences—the science of elementary arithmetic.'' The author discusses all the tendencies and movements of the last quarter of a century in arithmetic as sub-topics under the three main captions: progress in purpose and in topics, progress in the teaching art, and the textbook. In his characteristic style he does more than merely ''register'' and ''record;'' he evaluates in terms of his own experience and thinking and often in terms of his own prejudices. But this is the right and the duty of an authority.

Teachers looking for suggestions in the methods of teaching special topics in arithmetic or students desiring an expositional discussion of so-called methods and doctrines will have to read elsewhere. This essay is limited to another purpose. Teachers of arithmetic desirous of an authoritative but brief discussion of the main movements, influences, tendencies, changes, etc., in the teaching of arithmetic during the last twenty-five years will find it here. Educational experimentalists in the pedagogy of arithmetic will be interested in the statement of their failures as well as their helpful contributions in this field. Psychologists will feel grateful for the recognition of their contributions

to the better teaching of arithmetic. Teachers and students of other subjects taught in the elementary school will do well to read this statement of the progress in arithmetic in order to realize that this subject is far ahead of most of the elementary school subjects in the elimination of superfluous material and in the simplification and enrichment of the content of the present course. And yet many changes in the subject matter and teaching of arithmetic are still needed and are sure to come. This little volume serves as a powerful aid in directing attention to the present status of the teaching of arithmetic and to the need of continued care and improvement in the future teaching of the subject.

It is to be regretted that the publishers exercised the indiscretion of inserting a few pages of the author's textbooks in the publishers' supplement of fourteen pages to show the progress in arithmetic as exhibited in the pages of present day textbooks in arithmetic. Many readers will interpret this as a clever bit of advertising. However, the publishers did it, not the author.

Chicago Normal College. E. C. HINKLE

A CORRECTION

Note.—Corrections in an article entitled, "A Dutch Textbook of 1730," in the October number of THE MATHEMATICS TEACHER.

P. 341. L. 2 for "church" read "church.[6]"

P. 341 L. 7 for "teachers" read "teachers'."

P. 341 under illustration, L.4 for "printed by Jacob Goelet" read "printed for Jacob Goelet."

P. 344 L. 6 for "a which 'betekent tot'" read "a . which 'betekent tot'."

Since writing the above article, the author has found in Zenger's newspaper, "The New York Weekly Journal," for November 22, 1742, the solution of an algebraic problem proposed in an earlier issue. It is signed, "Opgelost door Petrus Venema," and this shows that Venema was still in New York or was still taking this newspaper in 1742.

NEWS NOTES

THE National Council of Teachers of Mathematics will hold its next annual meeting in Chicago on Saturday, February 23, 1924, preceding the national meeting of superintendents. A business session at 10:30 a. m., program at 2:30 p. m., banquet followed by program 6:30 p. m. A full program will be published in January.

The lists of men teachers of mathematics available for service in the City of Boston are practically exhausted. Qualifying examinations are held every year during the week after Christmas. Salaries for men begin at $1980, and increase in ten years to $3276, with chances for promotion. There is a contributory pension which insures retirement on three-quarters pay after thirty-eight years of service, and on full pay after forty-one years of service. Anybody desiring information about this opportunity may address the Chief Examiner, 15 Beacon Street, Boston.

THE Mathematics Section of the Illinois High School Conference met in Urbana, Nov. 24, 1923. The program included:

1. Oral Work in High School Algebra by Miss S. Helen Taylor, University High School, Urbana.
2. Relativity in Projective Geometry by Professor Henry Blumberg, University of Illinois.
3. The Purpose and Methods of Demonstrative Geometry by Professor W. W. Hart, University of Wisconsin.
4. An Experiment in Testing and Classifying Pupils in Beginning Algebra by C. M. Austin, Oak Park High School.

L. E. Mensenkamp of Freeport presided.

THE Association of Teachers of Mathematics in New England announced the following program for the Boston meeting on Dec. 8, 1923.

1. Newton's Junior High School Mathematics by Sarah E. Tracy.
2. Why not "Non-Euclidean" Algebras? by Professor Edward V. Huntington, Harvard.

3. Twenty years of the Association: Report of the Committee on Association History with Oral Addenda, by Professor W. H. Tyler, Massachusetts Institute of Technology.

4. Why We Teach Mathematics in High School, by Professor David Eugene Smith, Teachers College.

Mr. A. Harry Wheeler of the North High School, Worcester, is President of the Association.

THE first meeting of the Chicago Mathematics Club for the present year was held at the Central Y. M. C. A., 19 South La-Salle Street, on the evening of October 26. The officers are: President, Olice Winter, Harrison Technical; Secretary-Treasurer, Marx Holt, Crane College; Recording-Secretary, Edwin W. Schreiber, Proviso.

The question of sending a delegate to the National Council was discussed and the Club voted in the affirmative. Mr. Schreiber reported on the work of the Club for the season 1922-1923. The records show that 53 men representing 24 schools were in attendance sometime during the year. However, more than 20 schools in this vicinity were not represented and this year *special* effort will be made to *arouse* these schools. The Club enters its eleventh season with much enthusiasm as 39 were present at this first meeting.

The topic for discussion was "Mathematics Classroom Technique," with these divisions:

"In the University," Prof. G. W. Meyers, University of Chicago.

"In the Junior College," Mr. Marx Holt, Crane College.

"In the City High School," Mr. Arch. M. Allison, Lakeview.

"In the Suburban High School," Mr. Claude E. Kitch, Oak Park.

A Summary of Programs for 1922-'23 follows:

October 20, Review of Articles by Prof. Thorndike on Psychology of Algebra by Everett Owen, Oak Park.

November 17, The Slide Rule by W. W. Gorsline, Crane Junior College.

December 15, Field Problems, by M. W. Coultrap, Northwestern College, Naperville, and W. W. Gorsline, Crane Junior College.

January 19, Segregation in Mathematic Classes by F. A. Kahler, New Trier and C. M. Austin, Oak Park. Checking the Results of Classification in Nine First Year Algebra Classes by Means of the Hotz Algebra Scales, by Edwin W. Schreiber, Proviso.

February 16, Calculus in the High School, by J. M. Kinney, Crane Junior College.

March 16, Mathematics Used by the Actuary by J. Charles Seitz, Security Life Insurance Co., Chicago. Mathematics Used by the Construction Engineer, by John W. Musham, Condion Company, Chicago.

May 18, Deductive Sullogism by J. T. Johnson, Francis Parker. Report of Cleveland National Council Meeting by E. R. Breslich, University High. The Natural Science Approach to Geometry by J. O. Pyle, Harrison Tech.

Historical Note: The Chicago Mathematics Club has been in existence for ten years. The Club was started by C. M. Austin of Oak Park in January 1914. The Presidents with their term of office is given below: C. M. Austin, Oak Park, 1914-1916; John R. Clark, Chicago Normal, 1916-1918; W. W. Gorsline, Crane Jr. College, 1918-1920; M. J. Newell, Evanston, 1920-1921; H. C. Wright, University High, 1921-1922; Everett Owen, Oak Park, 1922-1923; Compiled by Edwin W. Schreiber, Recording-Secretary.

THE Mathematical Association of America sent the following communication to the High School Seniors of Ohio on May 29, 1923:

You may be going to college next year and, if so, you should now consider the advantages of electing courses in mathematics. The mathematics teachers of Ohio colleges hope that you have been successful in your algebra and geometry, and recommend that you seriously consider the continuation of mathematics in college for the following reasons:

1. It is desirable to know enough about mathematics to participate in conversation, to understand references, and, in general to have perspective on the matter when terms or ideas from mathematics come up. This field of study, in which law reigns more perfectly than in any other, has thrilled and still thrills some of the greatest men of all times.

2. Certain courses in mathematics are frequently set as prerequisite to getting into some of the other college subjects, and in many of the professional and technical schools. Trigonometry is indispensable for engineering, physics and astronomy, and is highly useful in many other fields. Some knowledge of higher mathematics is of great use in some of the professions and the study of mathematics is strongly recommended as a training for others, notably law, business, and medicine.

3. You may be in doubt what you will do after college. There is a good chance that you will need a mathematical background for your career. Fundamental courses should be begun early in college. Ignorance of mathematics may hinder your progress later and make you mediocre in your line.

4. The study of mathematics affords abundant opportunity for the exercise of numerous valuable mental traits. Ideals of accuracy, of careful consideration of data, of constantly checking conclusions, and of arranging work in logical order may result from proper instruction in this subject.

5. Mathematical problems present a challenge to our sporting instinct to conquer difficulties. Because you are not interested in high school mathematics, it does not necessarily follow that you will dislike this subject in college. There are many instances to the contrary.

6. *To those who like mathematics and have mathematical ability, this study offers opportunity for a scholarly career of study, teaching, and research.*

7. Mathematical information and power are peculiarly difficult to obtain without an instructor. Many other subjects may be more easily studied and read outside of college.

Very truly yours,

COMMITTEE.